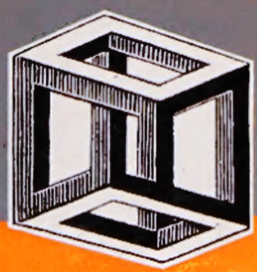
D. S. Mitrinović

Calculus of Residues



Tutorial Texts

TUTORIAL TEXT No. 4

CALCULUS OF RESIDUES

BY

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IN COOPERATION WITH

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No 1	ELEMENTARY INEQUALITIES	1964
No 2	FUNCTIONS OF A COMPLEX VARIABLE	1965
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FOREWORD

The problems contained in this series have been collected over many years with the aim of providing students and teachers with material, the search for which would otherwise occupy much valuable time. Hitherto this concentrated material has only been accessible to the very restricted public able to read Serbian *.

I greatly welcome the appearance of the Tutorial Texts based on my problem collection. Cooperation with colleagues in Australia and the U.S.A. and their initiative have made it possible. For the preparation of this Text, I wish to thank J. H. Michael.

D. S. MITRINOVIĆ

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* Zbornik matematičkih problema, Belgrade, Vol. I (1962), Vol. II (1960) Vol. III (1960), published in cooperation with D. Adamović, V. Devidé, D. Djoković, D. Mihailović, Z. Pop-Stojanović, S. Prešić, J. Ulčar.

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CHAPTER 0

INTRODUCTION

0.1 This book of problems deals with the application of the residue theorem to the evaluation of various types of integrals. The material has been classified into four chapters. Each of Chapters 1, 3 and 4 begins with a description of the particular type of problem under consideration and a discussion of the method used for solving it. The method is then illustrated by means of several worked examples and the chapter concludes with a set of examples to be solved by the reader. In contrast, Chapter 2 contains seven methods and hence is divided into eight sections, the final section consisting of miscellaneous examples.

Complete rigor is maintained throughout the book and the discussion of each method contains a precise statement of the conditions under which the method may be used. The theory of residues and the residue theorem are not discussed; it is felt that the best place for such a discussion is in a text on complex analysis, such as one of those listed below.

0.2 Use is made of the o and O notation. For the sake of completeness, it is defined here.

0.2.1 Let f and g be complex functions, defined on a set $N = \{z; 0 < |z - z_1| < \rho\}$.

We write

$$f(z) = o\{g(z)\}$$

as $z \to z_1$, if there exist a $\delta > 0$ (with $\delta \leqq \rho$) and a function $\varepsilon(z)$ such that

$$f(z) = \varepsilon(z) \cdot g(z)$$

for all $0 < |z - z_1| < \delta$ and

$$\varepsilon(z) \to 0$$

as $z \to z_1$.

We write

$$f(z) = O\{g(z)\}$$

as $z \to z_1$, if there exist a $\delta > 0$ (with $\delta \leqq \rho$) and a function $K(z)$ such that

$$f(z) = K(z) \cdot g(z)$$

for all $0 < |z-z_1| < \delta$ and $K(z)$ is bounded on the set

$$\{z;\ 0 < |z-z_1| < \delta\}.$$

0.2.2 Now suppose that f and g are complex functions, defined on a set $U = \{z;\ |z| > R\}$, where R is a positive constant.

We write

$$f(z) = o\{g(z)\}$$

as $z \to \infty$, if there exist an $S \geqq R$ and a function $\varepsilon(z)$ such that

$$f(z) = \varepsilon(z) \cdot g(z)$$

for all $|z| > S$ and

$$\varepsilon(z) \to 0$$

as $z \to \infty$.

We write

$$f(z) = O\{g(z)\}$$

as $z \to \infty$, if there exist an $S \geqq R$ and a function $K(z)$ such that

$$f(z) = K(z) \cdot g(z)$$

for all $|z| > S$ and $K(z)$ is bounded on the set $\{z;\ |z| > S\}$.

0.3 Use is also made of the following identities, which are easily verified. In each of these $z = x+iy$ and x, y are real.

0.3.1 $|e^{iz}| = e^{-y}$.

0.3.2 $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

0.3.3 $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

0.3.4 $$|\tan z|^2 = \frac{\sin^2 x + \sinh^2 y}{\cos^2 x + \sinh^2 y}.$$

0.3.5 $|\sinh z|^2 = \sinh^2 x + \sin^2 y$.

0.3.6 $|\cosh z|^2 = \sinh^2 x + \cos^2 y.$

0.3.7 $|\tanh z|^2 = \dfrac{\sinh^2 x + \sin^2 y}{\sinh^2 x + \cos^2 y}.$

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CHAPTER 1

DIRECT APPLICATION OF THE RESIDUE THEOREM

This chapter is devoted to the evaluation of integrals around simple, closed, positively oriented contours by means of the residue theorem, i.e., by finding the sum of the residues of the integrand at those singular points that lie inside the contour, and multiplying by $2\pi i$. Since there is essentially only one method, only a few worked examples are given. The chapter consists mainly of a list of unworked problems.

1.1 WORKED EXAMPLES.

1.1.1 Evaluate the integral

$$\oint_C \frac{z}{(z^2-1)^2(z^2+1)}\,dz,$$

where C is the positive orientation of the circle $\{z;\ |z-1|^2 = 3\}$.

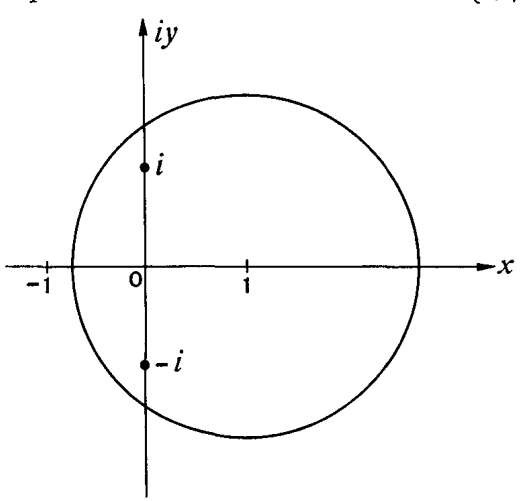

Fig. 1

SOLUTION: The function

$$f(z) = \frac{z}{(z^2-1)^2(z^2+1)}$$

has poles of order one at $\pm i$ and poles of order two at ± 1. Th of these singular points lie inside the circle, while the pole -1 is outside the circle (Fig. 1). The residue at i is

$$\lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{z}{(z^2-1)^2(z+i)} = \tfrac{1}{8}.$$

The residue at $-i$ is

$$\lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}\frac{z}{(z^2-1)^2(z-i)} = \tfrac{1}{8}.$$

The residue at 1 is

$$\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\{(z-1)^2 f(z)\} = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{z}{(z+1)^2(z^2+1)}\right\}$$

$$= \lim_{z\to 1}\frac{-3z^3-z^2-z+1}{(z+1)^3(z^2+1)^2} = -\tfrac{1}{8}.$$

Thus

$$\oint_C \frac{z}{(z^2-1)^2(z^2+1)}\,\mathrm{d}z = 2\pi i.\ \text{(sum of the residues)}$$

$$= 2\pi i(\tfrac{1}{8}+\tfrac{1}{8}-\tfrac{1}{8})$$

$$= \frac{\pi i}{4}.$$

1.1.2 Evaluate

$$\oint_C (\bar{z}-a)^{-1}(b-\bar{z})^{-1}(z^2+z^{-2})\mathrm{d}z, \qquad 0 < |a| < r < |b|,$$

where C is the positive orientation of the circle $\{z; |z| =$

SOLUTION: Since $z\bar{z} = r^2$, the integral is

$$\oint_C \frac{z^4+1}{(r^2-az)(bz-r^2)}\,\mathrm{d}z.$$

The function

$$f(z) = \frac{z^4+1}{(r^2-az)(bz-r^2)}$$

has poles of order one at r^2/a and r^2/b, but only the latter lies inside the circle. The residue at r^2/b is

$$\lim_{z\to r^2/b}\left(z-\frac{r^2}{b}\right)f(z) = \lim_{z\to r^2/b}\frac{z^4+1}{b(r^2-az)} = \frac{r^8+b^4}{b^4r^2(b-a)}.$$

Thus

$$\oint_C (\bar{z}-a)^{-1}(b-\bar{z})^{-1}(z^2+z^{-2})\mathrm{d}z = 2\pi i\frac{r^8+b^4}{b^4r^2(b-a)}.$$

1.1.3 Show that

$$\oint_C \cosh z\cot z\,\mathrm{d}z = 2\pi i\frac{\sinh(n+\frac{1}{2})\pi}{\sinh\frac{1}{2}\pi}, \qquad n = 0, 1, 2, \ldots,$$

where C is the positive orientation of the circle $\{z; |z| = (n+\frac{1}{2})\pi\}$.

SOLUTION: The function

$$f(z) = \cosh z\cot z$$

has singularities at the points $k\pi$, $k = 0, \pm1, \pm2, \ldots$. Since

$$f(z) = \frac{(-1)^k\cosh z\cos z}{\sin(z-k\pi)},$$

each singular point is a pole of order one. The residue at $k\pi$ is

$$\lim_{z\to k\pi}(z-k\pi)f(z) = \lim_{z\to k\pi}\frac{(-1)^k(z-k\pi)\cosh z\cos z}{\sin(z-k\pi)} = \cosh k\pi.$$

The point $k\pi$ lies inside the circle when $|k| \leqq n$. Thus

$$\oint_C \cosh z\cot z\,\mathrm{d}z = 2\pi i\sum_{k=-n}^{n}\cosh k\pi = 2\pi i\left(1+2\sum_{k=1}^{n}\cosh k\pi\right)$$

$$= 2\pi i\frac{\sinh(n+\frac{1}{2})\pi}{\sinh\frac{\pi}{2}}.$$

1.1.4 Evaluate the integral

$$\oint_C \frac{z^2}{e^{2\pi i z^3}-1}\,dz,$$

where C is the positive orientation of the circle $\{z;\ |z| = r\}$, n is a positive integer and $n < r^3 < n+1$.

SOLUTION: The function

$$f(z) = \frac{z^2}{e^{2\pi i z^3}-1}$$

has singular points, where

$$z^3 = k,\ k = 0, \pm 1, \pm 2, \ldots;$$

hence the singular points inside the contour are

$$0,\ \pm k^{\frac{1}{3}},\ \pm \alpha k^{\frac{1}{3}},\ \pm \alpha^2 k^{\frac{1}{3}}, \qquad k = 1, 2, \ldots, n,$$

where $\alpha = \dfrac{-1+i\sqrt{3}}{2}$.

If a is a complex number such that $e^{2\pi i a^3} = 1$, then

$$\lim_{z\to a} \frac{e^{2\pi i z^3}-1}{z-a} \text{ exists and } = \left[\frac{d}{dz}\left(e^{2\pi i z^3}\right)\right]_{z=a}$$

$$= \left[6\pi i z^2 e^{2\pi i z^3}\right]_{z=a} = 6\pi i a^2.$$

Hence, if $a \neq 0$,

$$\lim_{z\to a} \frac{(z-a)z^2}{e^{2\pi i z^3}-1} \text{ exists and } = \frac{1}{6\pi i};$$

therefore there is a pole of order one at a and the residue there is $1/(6\pi i)$.

There will be a pole of order one at 0, if

$$\lim_{z\to 0} \frac{z^3}{e^{2\pi i z^3}-1}$$

exists and the value of this limit will be the residue. But

$$\lim_{\zeta\to 0}\frac{e^{2\pi i\zeta}-1}{\zeta}\text{ exists and }=\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(e^{2\pi i\zeta}\right)\right]_{\zeta=0}$$

$$=\left[2\pi i e^{2\pi i\zeta}\right]_{\zeta=0}=2\pi i.$$

Then

$$\lim_{\zeta\to 0}\frac{\zeta}{e^{2\pi i\zeta}-1}\text{ exists and }=\frac{1}{2\pi i};$$

hence

$$\lim_{z\to 0}\frac{z^3}{e^{2\pi i z^3}-1}\text{ exists and }=\frac{1}{2\pi i}.$$

Thus, inside the contour, there is one singular point at which the residue is $1/(2\pi i)$ and $6n$ singular points at each of which the residue is $1/(6\pi i)$. Hence

$$\oint_C\frac{z^2}{e^{2\pi i z^3}-1}\,\mathrm{d}z=2\pi i\left(\frac{1}{2\pi i}+\frac{2n}{2\pi i}\right)=2n+1.$$

1.1.5 Evaluate the integral

$$J=\oint_C P(z)\left\{\exp\frac{1}{z}+\exp\frac{1}{z-1}+\ldots+\exp\frac{1}{z-k}\right\}\mathrm{d}z,$$

where P is a polynomial of degree k and C is the positive orientation of the circle $\{z;\,|z|=r\}$, $r\neq 1, 2, \ldots, k$.

SOLUTION: Let p be the non-negative integer such that $p<r<p+1$ when $r<k$, and let $p=k$ when $r>k$. Then

$$J=\sum_{j=0}^{p}\oint_C P(z)\exp\frac{1}{z-j}\,\mathrm{d}z.$$

The function

$$P(z)\exp\frac{1}{z-j}$$

has an essential singularity at j. Since

$$P(z)\exp\frac{1}{z-j}=\left\{P(j)+\frac{P'(j)}{1!}(z-j)+\frac{P''(j)}{2!}(z-j)^2+\ldots\right.$$
$$\left.\ldots+\frac{P^{(k)}(j)}{k!}(z-j)^k\right\}\left\{1+\frac{1}{1!(z-j)}+\frac{1}{2!(z-j)^2}+\ldots\right.$$

and since the residue is the coefficient of $1/(z-j)$ in this produc the residue is

$$\frac{P(j)}{0!1!}+\frac{P'(j)}{1!2!}+\frac{P''(j)}{2!3!}+\ldots+\frac{P^k(j)}{k!(k+1)!}.$$

Hence

$$\oint_C P(z)\exp\frac{1}{z-j}\,dz=2\pi i\sum_{m=0}^{k}\frac{P^{(m)}(j)}{m!(m+1)!}$$

and, consequently,

$$J=2\pi i\sum_{j=0}^{p}\sum_{m=0}^{k}\frac{P^{(m)}(j)}{m!(m+1)!}.$$

1.2 Examples

Evaluate the following integrals:

1.2.1

$$\oint_C\frac{dz}{(z^2+1)(z-1)^2},$$

where C is the positive orientation of the circle $\{z;\ |z-(1+i)|^2=2$

Answer: $-\dfrac{\pi i}{2}$.

1.2.2

$$\oint_C\frac{1}{(z^3+1)^2}\,dz,$$

where C is the positive orientation of the ellipse

$$\{x+iy;\ x^2+6xy+10y^2-4x+8y-3=0\}.$$

1.2.3

$$\oint_C \frac{e^{2iz}-5z}{z+2i}\,dz,$$

where C is the positive orientation of the circle $\{z;\ |z-\frac{1}{2}i| = 1\}$.

1.2.4

$$\oint_C \frac{\sin(a/z)}{z^2+b}\,dz, \qquad \oint_C \frac{\cos(a/z)}{z^2+b}\,dz,$$

where C is the positive orientation of the circle $\{z;\ |z| = r\}$, $r > 0$, and a, b are non-zero complex numbers with $|b| \neq r^2$.

1.2.5

$$\oint_C \frac{z^2}{z^4+4}\,dz,$$

where C is the positive orientation of the circle with centre 0 and radius 2.

1.2.6

$$\oint_C \frac{\cot z}{z(z-1)}\,dz,$$

where C is the positive orientation of the circle $\{z;\ |z| = 2\}$.

ANSWER: $2\pi i\,(\cot 1 - 1)$.

1.2.7

$$\oint_C \frac{dz}{1+z^4},$$

where C is the positive orientation of the ellipse

$$\{x+iy;\ x^2-xy+y^2+x+y = 0\}.$$

ANSWER: $(-1+i)\pi/(2\sqrt{2})$.

1.2.8

$$\oint_C \frac{\sin \pi z}{z^2(z^2-1)^4}\,dz,$$

where C is the positive orientation of the ellipse

$$\{z;\ |z|+|z-i| = k\}$$

and $k > 1$, $k \neq 1+\sqrt{2}$.

1.2.9

$$\oint_C \frac{z^2(2z-1)^2}{(z-2)^2}\,dz,$$

where C is the positive orientation of the circle

$$\{x+iy;\ x^2+y^2-4x+3 = 0\}.$$

ANSWER: $168\pi i$.

1.2.10

$$\oint_C \frac{dz}{z^p(1+z+z^2)},$$

where p is a positive integer and C is the positive orientation of the circle with centre 0 and radius $\frac{1}{2}$.

1.2.11

$$\oint_C \frac{z}{(e^z-1)^2(e^{-z}-1)^2}\,dz,$$

where C is the positive orientation of the circle with centre 0 and radius r and r is a positive real number which is not an integral multiple of 2π.

1.2.12

$$\oint_C \frac{\sin z}{1+z+z^2+z^3}\,dz,$$

where C is the positive orientation of the circle $\{z;\ |z| = 2\}$.

1.2.13

$$\oint_C \frac{dz}{(z-a)^n(z-b)^n},$$

where n is a positive integer, C is the positive orientation of the unit circle and a, b are complex numbers such that $|a| < 1 < |b|$.

ANSWER:

$$\frac{(-1)^{n-1}2\pi i(2n-2)!}{(n-1)!(n-1)!(a-b)^{2n-1}}.$$

1.2.14

$$\oint_C \frac{z^3}{z+1} e^{\frac{1}{z}}\,dz,$$

where C is the positive orientation of the circle $\{z; |z| = 2\}$.

1.2.15

$$\oint_C \frac{e^{zt}z}{(z+a)(z-b)^2}\,dz,$$

where a, b, t are complex numbers such that $|b| < 1 < |a|$ and C is the positive orientation of the unit circle.

1.2.16

$$\oint_C \frac{z^m}{e^{2\pi i z^n}-1}\,dz,$$

where m, n are positive integers, C is the positive orientation of the circle $\{z; |z| = r\}$, N is a positive integer and $N < r^n < N+1$.

1.2.17

$$\oint_C e^{-z}z^{-n-1}\,dz,$$

where n is a positive integer and C is the positive orientation of the unit circle.

ANSWER:

$$\frac{2(-1)^n\pi i}{n!}.$$

1.2.18

$$\oint_C \frac{z\,dz}{(z-1)(z-2)\dots(z-n)},$$

where C is the positive orientation of the circle $\{z; |z| = \frac{3}{2}\}$ and n is a positive integer.

ANSWER:

$$\frac{(-1)^{n-1}\,2\pi i}{(n-1)!}.$$

GENERALISATION.

$$\oint_C \frac{f(z)\mathrm{d}z}{(z-1)(z-2)\ldots(z-n)},$$

where n is a positive integer, C is the positive orientation of the circle $\{z;\ |z| = r\}$, N is a positive integer, $N < r < N+1$ and $f(z)$ is analytic everywhere.

1.2.19

$$\oint_C \tan \pi z \ \mathrm{d}z,$$

where C is the positive orientation of the circle with centre 0 and positive integral radius n.

1.2.20

$$\oint_C \frac{\mathrm{d}z}{\sinh z},$$

where C is the positive orientation of the circle $\{z;\ |z| = (n+\frac{1}{2})\pi\}$, n being a non-negative integer.

1.2.21

$$\oint_C \frac{1}{z} e^z \ \mathrm{d}z,$$

where C is the positive orientation of the loop of the curve $\{z;\ |e^z/z| = e\}$.

ANSWER: $2\pi i$.

1.2.22

$$\oint_C \left(\sum_{\mu=0}^{m} a_\mu z^{m-\mu}\right)\left(\sum_{\nu=0}^{k} b_\nu \exp\frac{1}{z-\nu}\right)\mathrm{d}z,$$

where C is the positive orientation of the circle with centre 0 and radius r and r is not an integer.

HINT: Compare with **1.1.5.**

1.2.23

$$\int_{C^+} \frac{z}{e^{2\pi i z^2}-1}\,dz,$$

where C^+ is the curve with parametric representation $re^{i\theta}$, $0 \leqq \theta \leqq \pi$, and $n < r^2 < n+1$, n being a positive integer.

HINT: Note that the integrand is an odd function.

1.2.24 Show that

$$\int_{C_1} \frac{z(2z^2-1)}{(z^2+1)\sin\left(\frac{\pi}{z^2}\right)}\,dz = 2\int_{C_2} \frac{(2-\zeta)}{\zeta^2(1+\zeta)\sin \pi\zeta}\,d\zeta,$$

where C_1 and C_2 denote the positive orientations of the circles $\{z; |z| = 2\}$, $\{\zeta; |\zeta| = \frac{1}{4}\}$.

Hence evaluate

$$\int_{C_1} \frac{z(2z^2-1)}{(z^2+1)\sin\left(\frac{\pi}{z^2}\right)}\,dz.$$

CHAPTER 2

INTEGRATION ALONG THE REAL AXIS

This chapter deals with the evaluation of integrals of the form $\int_{-\infty}^{\infty}\phi(x)\mathrm{d}x$ or $\int_{0}^{\infty}\phi(x)\mathrm{d}x$, where in most cases $\phi(x)$ is real (when x is real). Seven methods are listed and each is illustrated by several worked examples. Each set of worked examples is followed by examples to be worked by the reader using that particular method. The last section consists of a miscellaneous set of examples. It is felt that these final examples will be of greater value than those listed earlier, because they give the reader the additional mental exercise of deciding which method to use.

2.1 FIRST METHOD

This method is applicable to integrals of the type

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x,$$

where

(i) except for at most a finite number of singular points, none of which lies on the real axis, $f(z)$ is analytic on an open set containing the real axis and the upper half-plane and

(ii) there exists a real number $\alpha > 1$ such that $f(z) = O(|z|^{-\alpha})$ as $z \to \infty$ in the upper half-plane.

Condition (ii) implies that the integral $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ converges absolutely. This method is especially useful for evaluating integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x, \qquad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\cos x\,\mathrm{d}x, \qquad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\sin x\,\mathrm{d}x,$$

where P and Q are polynomials such that the degree of Q exceeds the degree of P by at least two and Q has no real zeros. When

applying the method to the last two integrals, let $f(z) = P(z)e^{iz}/Q(z)$ and use the relation

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos x \, \mathrm{d}x + i \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin x \, \mathrm{d}x.$$

WORKED EXAMPLES.

2.1.a Evaluate

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4}.$$

SOLUTION: Consider the integral

$$\int_C \frac{1}{1+z^4} \mathrm{d}z,$$

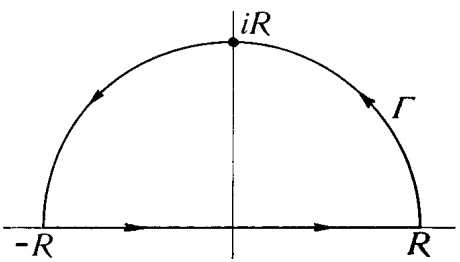

Fig. 2

where C is the contour shown in Fig. 2. The function $1/(1+z^4)$ has the singular points $z = e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$, the first two of which lie in the upper half-plane.
The residue at $e^{i\pi/4}$ is

$$\lim_{z \to e^{i\pi/4}} \frac{z - e^{i\pi/4}}{z^4+1} = \lim_{z \to e^{i\pi/4}} \frac{1}{(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})}$$

$$= -\frac{\sqrt{2}i(1+i)}{8}.$$

The residue at $e^{i3\pi/4}$ is

$$\lim_{z\to e^{i3\pi/4}} \frac{z-e^{i3\pi/4}}{z^4+1} = \lim_{z\to e^{i3\pi/4}} \frac{1}{(z-e^{i\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})}$$
$$= -\frac{\sqrt{2}i(1-i)}{8}.$$

By the residue theorem, when $R > 1$,

$$\int_C \frac{dz}{1+z^4} = 2\pi i\left[-\frac{\sqrt{2}i(1-i)}{8} - \frac{\sqrt{2}i(1+i)}{8}\right] = \frac{\pi\sqrt{2}}{2}.$$

But

$$\int_C \frac{dz}{1+z^4} = \int_{-R}^{R} \frac{dx}{1+x^4} + \int_\Gamma \frac{dz}{1+z^4},$$

where Γ denotes the semicircular part of the contour (Fig Since

$$\left|\int_\Gamma \frac{dz}{1+z^4}\right| \leq \frac{1}{R^4-1} \cdot \text{length of } \Gamma = \frac{\pi R}{R^4-1},$$

it follows that

$$\int_\Gamma \frac{dz}{1+z^4} \to 0 \qquad \text{as} \qquad R \to \infty.$$

Then

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{2},$$

i.e.,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{2}.$$

2.1.b Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2-2x+2}\,dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x^2-2x+2}\,dx.$$

SOLUTION: Consider the integral

$$\int_C \frac{e^{iz}}{z^2-2z+2}\,dz,$$

where C is the contour described in **2.1.a.**

The function $e^{iz}/(z^2-2z+2)$ has singularities where $z=1+i, 1-i$. However, $z = 1+i$ is the only singularity in the upper half-plane. Since this is a pole of order one, the residue is

$$\lim_{z\to 1+i} \frac{e^{iz}(z-1-i)}{z^2-2z+2} = \lim_{z\to 1+i} \frac{e^{iz}}{z-1+i} = -\frac{i}{2}e^{-1+i}.$$

By the residue theorem, if $R > \sqrt{2}$,

$$\int_C \frac{e^{iz}}{z^2-2z+2}\,dz = 2\pi i\left(-\frac{i}{2}e^{-1+i}\right) = \pi e^{-1+i}.$$

Now

$$\int_C \frac{e^{iz}}{z^2-2z+2}\,dz = \int_{-R}^{R} \frac{e^{ix}}{x^2-2x+2}\,dx + \int_\Gamma \frac{e^{iz}}{z^2-2z+2}\,dz.$$

But, when $z = x+iy$ lies on Γ, one has $y \geqq 0$ (cf. Fig. 2); hence one has, by **0.3.1**, $|e^{iz}| = e^{-y} \leqq 1$, so that

$$\left|\frac{e^{iz}}{z^2-2z+2}\right| \leqq \frac{1}{R^2-2R-2}.$$

Hence

$$\left|\int_\Gamma \frac{e^{iz}}{z^2-2z+2}\,dz\right| \leqq \frac{1}{R^2-2R-2}\cdot \text{length of } \Gamma$$
$$= \frac{\pi R}{R^2-2R-2}.$$

Therefore

$$\int_\Gamma \frac{e^{iz}}{z^2-2z+2}\,dz \to 0 \qquad \text{as} \qquad R\to\infty.$$

Thus

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x^2-2x+2}\,dx = \pi e^{-1+i},$$

i.e.,

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x^2-2x+2}\,dx = \frac{\pi}{e}(\cos 1 + i\sin 1);$$

hence

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2-2x+2}\,dx = \frac{\pi}{e}\cos 1, \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x^2-2x+2}\,dx = \frac{\pi}{e}\sin 1.$$

2.1.c Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1}\,dx.$$

SOLUTION: Since $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, consider the integral

$$\int_C \frac{1-e^{2iz}}{z^2+1}\,dz,$$

where C is the contour described in **2.1.a.**

The function $(1-e^{2iz})/(z^2+1)$ has singularities where $z = i, -i$ However the only singularity in the upper half-plane is $z = i$ a pole of order one, with residue

$$\lim_{z\to i} \frac{(1-e^{2iz})(z-i)}{z^2+1} = \lim_{z\to i} \frac{1-e^{2iz}}{z+i} = -\frac{i}{2}(1-e^{-2}).$$

By the residue theorem, when $R > 1$,

$$\int_C \frac{1-e^{2iz}}{z^2+1}\,dz = 2\pi i \cdot \left\{-\frac{i}{2}(1-e^{-2})\right\} = \pi(1-e^{-2}).$$

Now

$$\int_C \frac{1-e^{2iz}}{z^2+1}\,dz = \int_{-R}^{R} \frac{1-e^{2ix}}{x^2+1}\,dx + \int_\Gamma \frac{1-e^{2iz}}{z^2+1}\,dz.$$

But when $z = x+iy$ lies on Γ, one has $y \geqq 0$ (cf. Fig. 2); hence by **0.3.1**, $|e^{2iz}| = e^{-2y} \leqq 1$, so that

$$\left|\frac{1-e^{2iz}}{z^2+1}\right| \leqq \frac{2}{R^2-1}.$$

Hence

$$\left|\int_\Gamma \frac{1-e^{2iz}}{z^2+1}\,dz\right| \leqq \frac{2}{R^2-1} \cdot \text{length of } \Gamma$$

$$= \frac{2\pi R}{R^2-1}.$$

Therefore

$$\int_{\Gamma} \frac{1-e^{2iz}}{z^2+1}\,dz \to 0 \qquad \text{as} \qquad R \to \infty.$$

Thus

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{1-e^{2ix}}{x^2+1}\,dx = \pi(1-e^{-2}),$$

i.e.,

$$\int_{-\infty}^{\infty} \frac{(1-\cos 2x)-i\sin 2x}{x^2+1}\,dx = \pi(1-e^{-2});$$

hence

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1}\,dx = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos 2x}{x^2+1}\,dx = \frac{\pi}{2}(1-e^{-2}).$$

EXAMPLES

Evaluate the following integrals

2.1.1

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6}.$$

2.1.2

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2}\,dx, \qquad a>0,\ b>0.$$

ANSWER: $\pi e^{-ab}/b$.

2.1.3

$$\int_{-\infty}^{\infty} \frac{dx}{(x^4+1)(x^2+a^2)^2}, \qquad a>0.$$

ANSWER:

$$\pi\cdot\frac{5a^4+1+a^3\sqrt{2}\,(a^4-2a^2-1)}{2a^3(1+a^4)^2}.$$

2.1.4

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x(1+x^2)}\,dx.$$

HINT: The function f, defined by $f(z) = (1-e^{2iz})/z(1+z^2)$ when $z \neq 0$ and $f(0) = -2i$, is analytic at $z = 0$.

2.1.5

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\,dx.$$

2.1.6

$$\int_{-\infty}^{\infty} \frac{dx}{a^2+2b^2x^2+c^2x^4},$$

where a, b, $c > 0$ and $b^2-ac > 0$.

ANSWER:

$$\frac{\pi}{2^{\frac{3}{2}}\, a\sqrt{b^2-ac}}.$$

2.1.7

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^4+b^4}\,dx, \qquad a > 0,\ b > 0.$$

ANSWER:

$$\frac{\pi}{b^3}\, e^{-ab/\sqrt{2}} \sin\left(\frac{\pi}{4} + \frac{ab}{\sqrt{2}}\right).$$

2.1.8

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^3}\,dx.$$

ANSWER: $7\pi/8e$.

2.2 SECOND METHOD

This is a generalization of Method 1 to integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin^k x\,dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x)\cos^k x\,dx,$$

where

(i) except for at most a finite number of singular points, $f(z)$ is analytic on an open set containing the real axis and the upper half-plane,

(ii) there exists a real number $\alpha > 1$ such that $f(z) = O(|z|^{-\alpha})$ as $z \to \infty$ in the upper half-plane and

(iii) all the singular points of $f(z)$, that lie on the real axis, are poles of order k or less and occur at the zeros of $\sin x$ $(\cos x)$.

Conditions (ii) and (iii) imply that $\int_{-\infty}^{\infty} f(x)\sin^k x\,dx$ and $\int_{-\infty}^{\infty} f(x)\cos^k x\,dx$ converges absolutely.

WORKED EXAMPLES.

2.2.a Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x/2)}{x^2-1}\,dx.$$

SOLUTION: Consider the integral

$$\int_C \frac{e^{i\pi z/2}}{z^2-1}\,dz,$$

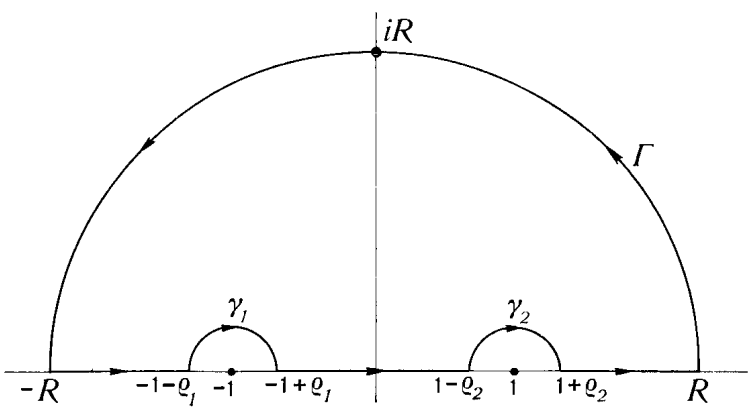

Fig. 3

where C is the contour shown in Fig. 3.

The function $e^{i\pi z/2}/(z^2-1)$ has singularities at the points $z = 1$ and $z = -1$, neither of which lie inside C; hence, by the residue theorem,

$$\int_C \frac{e^{i\pi z/2}\,dz}{z^2-1} = 0.$$

However,

$$\int_C \frac{e^{i\pi z/2}}{z^2-1}\,dz = \int_{-R}^{-1-\rho_1} \frac{e^{i\pi x/2}}{x^2-1}\,dx + \int_{-1+\rho_1}^{1-\rho} \frac{e^{i\pi x/2}}{x^2-1}\,dx$$
$$+ \int_{1+\rho_2}^{R} \frac{e^{i\pi x/2}}{x^2-1}\,dx + \int_\Gamma \frac{e^{i\pi z/2}}{z^2-1}\,dz + \int_{\gamma_1} \frac{e^{i\pi z/2}}{z^2-1}\,dz + \int_{\gamma_2} \frac{e^{i\pi z/2}}{z^2-1}\,dz,$$

where Γ denotes the large semi-circular part and γ_1, γ_2 the two small semi-circular parts of the contour C (Fig. 3).

When $z = x+iy$ lies on Γ, one has $y \geqq 0$; hence, by **0.3.1**, $|e^{i\pi z/2}| = e^{-\pi y/2} \leqq 1$, so that

$$\left|\frac{e^{i\pi z/2}}{z^2-1}\right| \leqq \frac{1}{R^2-1},$$

and hence

$$\left|\int_\Gamma \frac{e^{i\pi z/2}}{z^2-1}\,dz\right| \leqq \frac{1}{R^2-1} \cdot \text{length of } \Gamma = \frac{\pi R}{R^2-1}.$$

Therefore

$$\int_\Gamma \frac{e^{i\pi z/2}}{z^2-1}\,dz \to 0 \qquad \text{as} \qquad R \to \infty.$$

The residue of $e^{i\pi z/2}/(z^2-1)$ at $z = -1$ is

$$\lim_{z\to -1} \frac{e^{i\pi z/2}(z+1)}{z^2-1} = \lim_{z\to -1} \frac{e^{i\pi z/2}}{z-1} = \frac{i}{2},$$

and hence, in a neighbourhood of -1,

$$\frac{e^{i\pi z/2}}{z^2-1} = \frac{i}{2(z+1)} + g(z),$$

where $g(z)$ is analytic at $z = -1$. Hence, if ρ_1 is small,

$$\int_{\gamma_1} \frac{e^{i\pi z/2}}{z^2-1}\,dz = \frac{i}{2}\int_{\gamma_1} \frac{dz}{z+1} + \int_{\gamma_1} g(z)\,dz.$$

But

$$\int_{\gamma_1} \frac{dz}{z+1} = -i\int_0^\pi d\theta = -\pi i$$

and, since $|g(z)| \leqq K = \text{const.}$ in a neighbourhood of -1,

$$\left| \int_{\gamma_1} g(z)\mathrm{d}z \right| \leqq K \cdot \text{length of } \gamma_1 = K\pi\rho_1,$$

so that

$$\int_{\gamma_1} g(z)\mathrm{d}z \to 0$$

as $\rho_1 \to 0$. Thus,

$$\int_{\gamma_1} \frac{e^{i\pi z/2}}{z^2-1}\,\mathrm{d}z \to \frac{\pi}{2} \quad \text{as} \quad \rho_1 \to 0.$$

Similarly, the residue of $e^{i\pi z/2}/(z^2-1)$ at $z = 1$ is

$$\lim_{z\to 1} \frac{e^{i\pi z/2}(z-1)}{z^2-1} = \lim_{z\to 1} \frac{e^{i\pi z/2}}{z+1} = \frac{i}{2};$$

hence, in a neighbourhood of 1,

$$\frac{e^{i\pi z/2}}{z^2-1} = \frac{i}{2(z-1)} + h(z),$$

where $h(z)$ is analytic at $z = 1$. Hence

$$\int_{\gamma_2} \frac{e^{i\pi z/2}}{z^2-1}\,\mathrm{d}z \to \frac{\pi}{2}$$

as $\rho_2 \to 0$.

Thus,

$$\int_{-R}^{-1-\rho_1} \frac{e^{i\pi x/2}}{x^2-1}\,\mathrm{d}x + \int_{-1+\rho_1}^{1-\rho_2} \frac{e^{i\pi x/2}}{x^2-1}\,\mathrm{d}x + \int_{1+\rho_2}^{R} \frac{e^{i\pi x/2}}{x^2-1}\,\mathrm{d}x \to -\pi$$

as $R \to \infty$, $\rho_1 \to 0$ and $\rho_2 \to 0$, i.e.,

$$\int_{-R}^{-1-\rho_1} \frac{\cos(\pi x/2)+i\sin(\pi x/2)}{x^2-1}\,\mathrm{d}x + \int_{-1+\rho_1}^{1-\rho_2} \frac{\cos(\pi x/2)+i\sin(\pi x/2}{x^2-1}$$

$$+ \int_{1+\rho_2}^{R} \frac{\cos(\pi x/2)+i\sin(\pi x/2)}{x^2-1}\,\mathrm{d}x \to -\pi.$$

By equating real parts, one finds

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x/2)}{x^2-1}\,dx = -\pi.$$

2.2.b Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx.$$

SOLUTION: Since $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$, consider the tegral

$$\int_C \frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{3iz}}{z^3}\,dz,$$

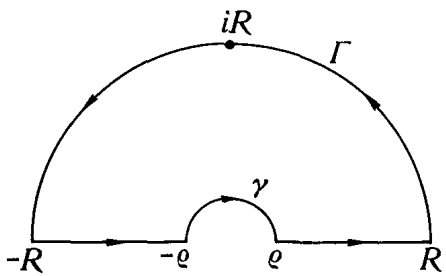

Fig. 4

where C is the contour shown in Fig. 4. The function

$$\frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{3iz}}{z^3}$$

has only one singularity, and that is at $z = 0$. Hence

$$\int_C \frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{3iz}}{z^3}\,dz = 0.$$

Now

$$\int_C \frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{3iz}}{z^3}\,dz = \int_{-R}^{-\rho} \frac{\frac{3}{4}e^{ix} - \frac{1}{4}e^{3ix}}{x^3}\,dx + \int_{\rho}^{R} \frac{\frac{3}{4}e^{ix} - \frac{1}{4}e^{3ix}}{x^3}\,dx$$
$$+ \int_{\Gamma} \frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{3iz}}{z^3}\,dz + \int_{\gamma} \frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{3iz}}{z^3}\,dz,$$

where Γ denotes the large semi-circular part and γ the small semi-circular part of the contour C (Fig. 4).

But when $z = x+iy$ lies on Γ, one has $y \geqq 0$; hence, by **0.3.1**, $|e^{iz}| = e^{-y} \leqq 1$ and $|e^{3iz}| = e^{-3y} \leqq 1$, so that

$$\left|\frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{3iz}}{z^3}\right| < \frac{1}{R^3}.$$

Hence

$$\left|\int_\Gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{3iz}}{z^3}\,dz\right| \leqq \frac{1}{R^3}\cdot \text{length of } \Gamma = \frac{\pi R}{R^3}.$$

Therefore

$$\int_\Gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{3iz}}{z^3}\,dz \to 0$$

as $R \to \infty$. Since

$$\begin{aligned}\frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{3iz}}{z^3} &= \frac{1}{z^3}\left[\frac{3}{4}\left(1+\frac{iz}{1!}-\frac{z^2}{2!}-\frac{iz^3}{3!}+\dots\right)\right.\\ &\quad\left.-\frac{1}{4}\left(1+\frac{3iz}{1!}-\frac{9z^2}{2!}-\frac{27iz^3}{3!}+\dots\right)\right]\\ &= \frac{1}{2z^3}+\frac{3}{4z}+g(z),\end{aligned}$$

where $g(z)$ is analytic, it follows that

$$\int_\gamma \frac{\frac{3}{4}e^{iz}-\frac{1}{4}e^{3iz}}{z^3}\,dz = \frac{1}{2}\int_\gamma \frac{dz}{z^3}+\frac{3}{4}\int_\gamma \frac{dz}{z}+\int_\gamma g(z)dz.$$

But

$$\int_\gamma \frac{dz}{z^3} = -\frac{1}{\rho^3}\int_0^\pi e^{-3i\theta}\rho i e^{i\theta}\,d\theta = -\frac{i}{\rho^2}\int_0^\pi (\cos 2\theta - i\sin 2\theta)\,d\theta = 0,$$

$$\int_\gamma \frac{dz}{z} = -i\int_0^\pi d\theta = -\pi i.$$

Since $|g(z)| \leqq K = \text{const.}$ in a neighbourhood of 0, one has, for small ρ,

$$\left|\int_\gamma g(z)\mathrm{d}z\right| \leqq K \cdot \text{length of } \gamma = K\pi\rho,$$

so that

$$\int_\gamma g(z)\mathrm{d}z \to 0$$

as $\rho \to 0$.

Thus,

$$\int_{-R}^{-\rho} \frac{\frac{3}{4}e^{ix} - \frac{1}{4}e^{3ix}}{x^3}\,\mathrm{d}x + \int_\rho^R \frac{\frac{3}{4}e^{ix} - \frac{1}{4}e^{3ix}}{x^3}\,\mathrm{d}x \to \tfrac{3}{4}\pi i$$

as $\rho \to 0$ and $R \to \infty$. By equating imaginary parts, one fin

$$\int_{-R}^{-\rho} \frac{\frac{3}{4}\sin x - \frac{1}{4}\sin 3x}{x^3}\,\mathrm{d}x + \int_\rho^R \frac{\frac{3}{4}\sin x - \frac{1}{4}\sin 3x}{x^3}\,\mathrm{d}x \to \tfrac{3}{4}\pi$$

as $\rho \to 0$ and $R \to \infty$, i.e.,

$$\int_{-R}^{-\rho} \frac{\sin^3 x}{x^3}\,\mathrm{d}x + \int_\rho^R \frac{\sin^3 x}{x^3}\,\mathrm{d}x \to \tfrac{3}{4}\pi$$

as $\rho \to 0$ and $R \to \infty$. Thus

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,\mathrm{d}x = \tfrac{3}{4}\pi.$$

EXAMPLES

Evaluate the following integrals:

2.2.1

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)}\,\mathrm{d}x, \qquad a > 0.$$

ANSWER: $\pi(1-e^{-a})/a^2$.

2.2.2

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2(1+x^2)}\,\mathrm{d}x.$$

ANSWER: $\pi(1+e^{-2})/2$.

2.3 THIRD METHOD

This method is applicable to integrals

$$\int_0^\infty f(x)\,\mathrm{d}x,$$

where $f(x)$ is an integrand of one of the types considered Methods 1 and 2 and $f(x)$ is also an even function. One has

$$\int_0^\infty f(x)\mathrm{d}x = \tfrac{1}{2}\int_{-\infty}^\infty f(x)\mathrm{d}x$$

and the integral on the right-hand side can be evaluated Method 1 or 2.

WORKED EXAMPLES.

2.3.a Evaluate

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^4}.$$

SOLUTION: Since $\dfrac{1}{1+x^4}$ is even,

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^4} = \tfrac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{d}x}{1+x^4}$$

and, by Example **2.1.a**,

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^4} = \frac{\pi\sqrt{2}}{4}.$$

2.3.b Evaluate

$$\int_0^\infty \frac{\cos(\pi x/2)}{x^2-1}\,\mathrm{d}x.$$

SOLUTION: Since $\dfrac{\cos(\pi x/2)}{x^2-1}$ is even,

$$\int_0^\infty \frac{\cos(\pi x/2)}{x^2-1}\,\mathrm{d}x = \tfrac{1}{2}\int_{-\infty}^\infty \frac{\cos(\pi x/2)}{x^2-1}\,\mathrm{d}x,$$

and, by Example **2.2.a**,

$$\int_0^\infty \frac{\cos(\pi x/2)}{x^2-1}\,dx = -\frac{\pi}{2}.$$

EXAMPLES

Evaluate the integrals:

2.3.1

$$\int_0^\infty \frac{dx}{(1+x^2)^2}.$$

2.3.2

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$$

2.4 FOURTH METHOD.

This applies to integrals

$$\int_{-\infty}^{\infty} f(x)\sin x\,dx = 2\int_0 f(x)\sin x\,dx,$$

where

(i) $f(x)\sin x$ is an even function of x *),

(ii) except for at most a finite number of singular points, $f(z)$ is analytic on an open set containing the real axis and the upper half-plane,

(iii) all singular points of $f(z)$ that lie on the real axis are poles of order one and occur at the zeros of $\sin x$,

---

*) Method 4 could be applied to integrals of the form $\int_{-\infty}^{\infty} f(x)\sin x\,dx$, when $f(x)\sin x$ is not an even function, but all one would obtain is
(i) a proof that

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\sin x\,dx$$

exists and
(ii) the value of this limit.
Since the existence of this limit does not imply the convergence of $\int_{-\infty}^{\infty} f(x)\sin x$ in the usual sense, it is questionable whether one is entitled to call this limit the integral $\int_{-\infty}^{\infty} f(x)\sin x\,dx$.

(iv)

$$f(z) = O(|z|^{-1})$$

as $z \to \infty$ in the upper half plane.

The convergence of the integral $\int_{-\infty}^{\infty} f(x)\sin x\,\mathrm{d}x = 2\int_0^{\infty} f(x)\sin x\,\mathrm{d}x$ may only be conditional. It will be seen that the integral does converge, when it is shown in a given example that $\lim_{R\to\infty}\int_{-R}^{R} f(x)\sin x\,\mathrm{d}x$ exists.

When using Method 4 we need Jordan's inequality

$$\int_0^{\pi} e^{-a\sin\theta}\,\mathrm{d}\theta \leqq \frac{\pi}{a}\,(1-e^{-a}), \qquad a > 0, \tag{2.4.1}$$

This inequality can be obtained as follows. The function

$$g(\theta) = \sin\theta - \theta\cos\theta$$

has a non-negative derivative when $0 \leqq \theta \leqq \pi/2$, so that, since $g(0) = 0$,

$$\sin\theta - \theta\cos\theta \geqq 0$$

when $0 \leqq \theta \leqq \pi/2$. Hence

$$\theta^{-2}(\theta\cos\theta - \sin\theta) \leqq 0$$

for $0 < \theta \leqq \pi/2$ and, since the expression on the left-hand side is the derivative of the function

$$f(\theta) = \frac{\sin\theta}{\theta},$$

it follows that f is decreasing for $0 < \theta \leqq \pi/2$. Thus

$$\frac{\sin\theta}{\theta} \geqq f\left(\frac{\pi}{2}\right) = \frac{2}{\pi},$$

i.e., $\sin\theta \geqq 2\theta/\pi$ for $0 < \theta \leqq \pi/2$. Therefore

$$e^{-a\sin\theta} \leqq e^{-a2\theta/\pi},$$

so that

$$\int_0^{\pi} e^{-a\sin\theta}\,\mathrm{d}\theta = 2\int_0^{\pi/2} e^{-a\sin\theta}\,\mathrm{d}\theta \leqq 2\int_0^{\pi/2} e^{-a2\theta/\pi}\,\mathrm{d}\theta = \frac{\pi}{a}\,(1-e^{-a}).$$

WORKED EXAMPLES.

2.4.a Evaluate

$$\int_0^\infty \frac{x \sin x}{x^2+1}\,dx.$$

SOLUTION: Consider the integral

$$\int_C \frac{ze^{iz}}{z^2+1}\,dz,$$

where C is the contour shown in Fig. 5.

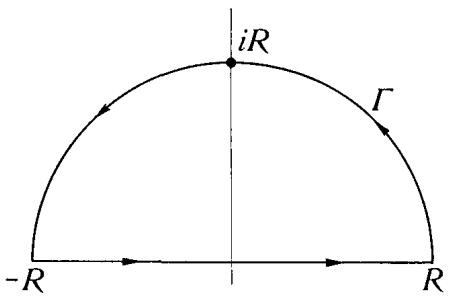

Fig. 5

The function

$$\frac{ze^{iz}}{z^2+1}$$

has poles of order one for $z = i$ and $z = -i$, where only former lies in the upper half-plane. The residue at i is

$$\lim_{z\to i} \frac{ze^{iz}(z-i)}{z^2+1} = \lim_{z\to i} \frac{ze^{iz}}{z+i} = \frac{1}{2e}.$$

By the residue theorem,

$$\int_C \frac{ze^{iz}}{z^2+1}\,dz = \frac{\pi i}{e},$$

when $R > 1$. However,

$$\int_C \frac{ze^{iz}}{z^2+1}\,dz = \int_{-R}^{R} \frac{xe^{ix}}{x^2+1}\,dx + \int_\Gamma \frac{ze^{iz}}{z^2+1}\,dz,$$

where Γ denotes the semi-circular part of the contour C (Fig. By making the substitution $z = R(\cos\theta + i\sin\theta)$, one fi

$$\int_\Gamma \frac{ze^{iz}}{z^2+1}\,dz = R\int_0^\pi \frac{ze^{iR\cos\theta - R\sin\theta}}{z^2+1}(-\sin\theta + i\cos\theta)d\theta;$$

hence

$$\left|\int_\Gamma \frac{ze^{iz}}{z^2+1}\,dz\right| \leqq R\int_0^\pi \left|\frac{ze^{iR\cos\theta - R\sin\theta}}{z^2+1}(-\sin\theta + i\cos\theta)\right| d\theta$$

$$\leqq \frac{R^2}{R^2-1}\int_0^\pi e^{-R\sin\theta}\,d\theta,$$

so that, by Jordan's inequality (2.4.1),

$$\left|\int_\Gamma \frac{ze^{iz}}{z^2+1}\,dz\right| \leqq \frac{R^2}{R^2-1}\cdot\frac{\pi}{R}(1-e^{-R}).$$

Therefore

$$\int_\Gamma \frac{ze^{iz}}{z^2+1}\,dz \to 0,$$

as $R \to \infty$, and hence

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{xe^{ix}}{x^2+1}\,dx = \frac{\pi i}{e},$$

i.e.,

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x(\cos x + i\sin x)}{x^2+1}\,dx = \frac{\pi i}{e}.$$

By equating imaginary parts, one obtains

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x\sin x}{x^2+1} = \frac{\pi}{e}$$

and, since $(x\sin x)/(x^2+1)$ is even,

$$\lim_{R\to\infty}\int_0^R \frac{x\sin x}{x^2+1}\,dx = \frac{\pi}{2e},$$

i.e.,

$$\int_0^\infty \frac{x\sin x}{x^2+1}\,dx = \frac{\pi}{2e}.$$

2.4.b Evaluate

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

SOLUTION: Consider the integral

$$\int_C \frac{e^{iz}}{z}\,dz,$$

where C is the contour shown in Fig. 6.

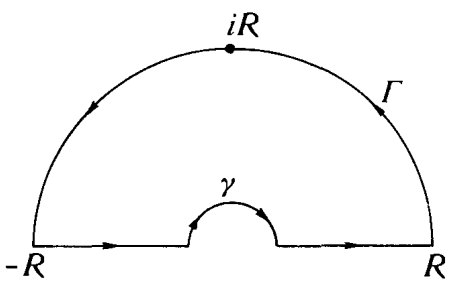

Fig. 6

The function e^{iz}/z has a singularity where $z = 0$. Since this poi lies outside C, one has, by the residue theorem,

$$\int_C \frac{e^{iz}}{z}\,dz = 0.$$

However,

$$\int_C \frac{e^{iz}}{z}\,dz = \int_{-R}^{-\rho} \frac{e^{ix}}{x}\,dx + \int_\rho^R \frac{e^{ix}}{x}\,dx + \int_\Gamma \frac{e^{iz}}{z}\,dz + \int_\gamma \frac{e^{iz}}{z}\,dz,$$

where Γ denotes the large semi-circular part and γ the sm semi-circular part of the contour C (Fig. 6). By making t substitution $z = R(\cos\theta + i\sin\theta)$, one finds that

$$\int_\Gamma \frac{e^{iz}}{z}\,dz = R\int_0^\pi \frac{e^{iR\cos\theta - R\sin\theta}}{z}\,(-\sin\theta + i\cos\theta)d\theta;$$

hence

$$\left|\int_\Gamma \frac{e^{iz}}{z}\,dz\right| \leqq R\int_0^\pi \left|\frac{e^{iR\cos\theta - R\sin\theta}}{z}(-\sin\theta + i\cos\theta)\right| d\theta$$
$$= \int_0^\pi e^{-R\sin\theta}\,d\theta$$

and, by Jordan's inequality (**2.4.1**),

$$\left|\int_\Gamma \frac{e^{iz}}{z}\,dz\right| \leqq \frac{\pi}{R}(1-e^{-R}),$$

so that

$$\int_\Gamma \frac{e^{iz}}{z}\,dz \to 0$$

as $R \to \infty$. The residue of e^{iz}/z at $z = 0$ is

$$\lim_{z\to 0} \frac{e^{iz}z}{z} = 1;$$

hence, in a neighbourhood of 0,

$$\frac{e^{iz}}{z} = \frac{1}{z} + g(z),$$

where $g(z)$ is analytic at $z = 0$. Thus, if ρ is small,

$$\int_\gamma \frac{e^{iz}}{z}\,dz = \int_\gamma \frac{dz}{z} + \int_\gamma g(z)\,dz.$$

However,

$$\int_\gamma \frac{dz}{z} = -i\int_0^\pi d\theta = -\pi i$$

and, since $|g(z)| \leqq K = \text{const.}$ in a neighbourhood of 0,

$$\left|\int_\gamma g(z)\,dz\right| \leqq K \cdot \text{length of } \gamma = K\pi\rho,$$

so that

$$\int_\gamma g(z)\,dz \to 0$$

as $\rho \to 0$. Therefore

$$\int_\gamma \frac{e^{iz}}{z} \to -\pi i$$

as $\rho \to 0$. Thus,

$$\int_{-R}^{-\rho} \frac{e^{ix}}{x}\,dx + \int_\rho^R \frac{e^{ix}}{x}\,dx \to \pi i$$

as $\rho \to 0$ and $R \to \infty$. By equating imaginary parts, one finds t

$$\int_{-R}^{-\rho} \frac{\sin x}{x}\,dx + \int_\rho^R \frac{\sin x}{x}\,dx \to \pi$$

and, since $\sin x/x$ is even,

$$\int_\rho^R \frac{\sin x}{x}\,dx \to \frac{\pi}{2},$$

i.e.,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

EXAMPLES

Evaluate the following integrals:

2.4.1

$$\int_0^\infty \frac{x^3 \sin x}{(x^2+1)^2}\,dx.$$

2.4.2

$$\int_0^\infty \frac{x^2-b^2}{x^2+b^2} \cdot \frac{\sin ax}{x}\,dx, \qquad a > 0, b > 0.$$

ANSWER: $\pi(e^{-ab}-\frac{1}{2})$.

2.4.3

$$\int_{-\infty}^{+\infty} \frac{a\cos x + x \sin x}{x^2+a^2}\,dx, \qquad a > 0.$$

ANSWER: $2\pi e^{-a}$.

2.5 Fifth method

This method is applicable to integrals of the form

$$\int_0^\infty x^\alpha f(x)\mathrm{d}x,$$

where

(i) $-1 < \alpha < 0$,

(ii) $f(z)$ is analytic on the entire complex plane except for at most a finite number of singular points, none of which lies on the non-negative part of the real axis and

(iii) $f(z) = O(z^{-1})$ as $z \to \infty$.

Condition (i) and the continuity of $f(z)$ at $z = 0$ imply that $\int_0^1 x^\alpha f(x)\mathrm{d}x$ converges absolutely, while conditions (i) and (iii) imply that $\int_1^\infty x^\alpha f(x)\mathrm{d}x$ converges absolutely.

Condition (iii) can be replaced by the condition

$$\text{(A)} \qquad f(z) = o(|z|^{-\alpha-1}) \qquad \text{as} \qquad z \to \infty,$$

which is apparently weaker. However, it can be shown that (ii) and (A) imply (iii); hence one might just as well use the condition (iii), which is easier to apply.

Method 5 serves mainly for the evaluation of integrals of the form

$$\int_0^\infty x^\alpha \frac{P(x)}{Q(x)}\,\mathrm{d}x,$$

where $-1 < \alpha < 0$ and $P(x)$, $Q(x)$ are polynomials such that the degree of Q exceeds that of P by at least one. The method cannot be applied to integrals of the form

$$\int_0^\infty x^\alpha \frac{P(x)}{Q(x)} \sin x\,\mathrm{d}x, \tag{2.5.1}$$

because the function $P(z)e^{iz}/Q(z)$ is not $O(z^{-1})$ as $z \to \infty$.

This remark is not in conflict with the proof in Method 1 of the boundedness of $|e^{iz}|$, because there one was concerned only with the upper half-plane, but in Method 5 one is concerned with the entire complex plane. When $z = -iy$, with y real and positive, one has $e^{iz} = e^y = e^{|z|}$, and hence one never has $e^{iz} = O(|z|^n)$, even when n is large and positive.

Since Method 5 cannot be applied to integrals of the form (2.5.1), no attempt will be made to generalise it to the case where $f(z)$ has poles on the non-negative part of the real axis.

WORKED EXAMPLES.

2.5.a Evaluate

$$\int_0^\infty \frac{x^{-\frac{1}{3}}}{1+x^2}\,\mathrm{d}x.$$

SOLUTION: Consider the two integrals

$$\int_{C_1} f_1(z)\mathrm{d}z, \quad \int_{C_2} f_2(z)\mathrm{d}z,$$

where C_1 and C_2 are the two contours shown in Fig. 7, and

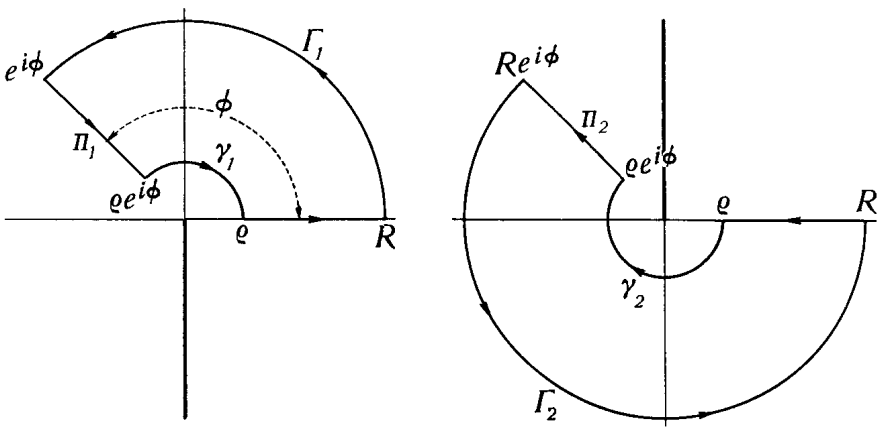

Fig. 7

$$f_1(z) = \frac{z^{-\frac{1}{3}}}{1+z^2} = \frac{e^{-\frac{1}{3}\ln z}}{1+z^2}, \qquad -\frac{\pi}{2} < \mathscr{I}m \ln z < \frac{3\pi}{2},$$

$$f_2(z) = \frac{z^{-\frac{1}{3}}}{1+z^2} = \frac{e^{-\frac{1}{3}\ln z}}{1+z^2}, \qquad \frac{\pi}{2} < \mathscr{I}m \ln z < \frac{5\pi}{2},$$

and the angle ϕ is chosen so that the half-line from 0 through $Re^{i\phi}$ does not pass through any singular points of the function $1/(1+z^2)$.

The function $f_1(z)$ has one singularity inside C_1, namely $z = i$, and the residue is

$$\lim_{z\to i} \frac{z^{-\frac{1}{3}}(z-i)}{z^2+1} = \frac{i^{-\frac{1}{3}}}{2i} = -\frac{i}{2}\, e^{-i\pi/6}.$$

Hence, by the residue theorem,

$$\int_{C_1} f_1(z)\, dz = 2\pi i\left(-\frac{i}{2}\, e^{-i\pi/6}\right) = \pi e^{-i\pi/6} = \frac{\pi}{2}\,(\sqrt{3}-i).$$

The function $f_2(z)$ has one singularity inside C_2, namely at $z =$ and the residue is

$$\lim_{z\to -i} \frac{z^{-\frac{1}{3}}(z+i)}{z^2+1} = \frac{(-i)^{-\frac{1}{3}}}{-2i} = \frac{i}{2}\, e^{-i\pi/2}.$$

Hence, by the residue theorem,

$$\int_{C_2} f_2(z)dz = 2\pi i\left(\frac{i}{2}\, e^{-i\pi/2}\right) = -\pi\, e^{-i\pi/2} = \pi i.$$

Since $\int_1 f_1(z)dz + \int_2 f_2(z)dz = 0$, it follows that

$$\int_{C_1} f_1(z)dz + \int_{C_2} f_2(z)dz = \int_\rho^R f_1(x)dx - \int_\rho^R f_2(x)dx + \int_{\Gamma_1} f_1(z)dz$$
$$+ \int_{\gamma_1} f_1(z)dz + \int_{\Gamma_2} f_2(z)dz + \int_{\gamma_2} f_2(z)dz,$$

where Γ_i is the large circular arc and γ_i the small circular of the contour C_i (Fig. 7).

When z lies on Γ_j

$$|f_j(z)| = \left|\frac{z^{-\frac{1}{3}}}{1+z^2}\right| \leqq \frac{R^{-\frac{1}{3}}}{R^2-1};$$

hence

$$\left|\int_{\Gamma_j} f_j(z)dz\right| \leqq \frac{R^{-1/3}}{R^2-1} \cdot \text{length of } \Gamma_j < \frac{R^{-1/3}}{R^2-1} \cdot 2\pi R.$$

Thus,

$$\int_{\Gamma_j} f_j(z)dz \to 0$$

as $R \to \infty$. When z lies on γ_j and $\rho < 1/\sqrt{2}$,

$$|f_j(z)| = \left| \frac{z^{-\frac{1}{3}}}{1+z^2} \right| \leqq 2\rho^{-\frac{1}{3}};$$

hence

$$\left| \int_{\gamma_j} f_j(z)\mathrm{d}z \right| \leqq 2\rho^{-\frac{1}{3}} \cdot \text{length of } \gamma_j < 2\rho^{\frac{1}{3}} \cdot 2\pi\rho.$$

Thus,

$$\int_{\gamma_j} f_j(z)\mathrm{d}z \to 0$$

as $\rho \to 0$.

It follows that

$$\int_\rho^R f_1(x)\mathrm{d}x - \int_\rho^R f_2(x)\mathrm{d}x \to \frac{\pi}{2}(\sqrt{3}-i)+\pi i = \frac{\pi}{2}(\sqrt{3}+i)$$

as $\rho \to 0$ and $R \to \infty$. However,

$$\int_\rho^R \{f_1(x)-f_2(x)\}\mathrm{d}x = \int_\rho^R \frac{1}{1+x^2}\left\{e^{-\frac{1}{3}\ln x} - e^{-\frac{1}{3}(\ln x+2\pi i)}\right\}\mathrm{d}x$$

$$= \int_\rho^R \frac{1}{1+x^2}\left\{x^{-\frac{1}{3}} - x^{-\frac{1}{3}}e^{-\frac{2\pi i}{3}}\right\}\mathrm{d}x = \left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)\int_\rho^R \frac{x^{-\frac{1}{3}}}{1+x^2}\mathrm{d}x,$$

so that

$$\int_\rho^R \frac{x^{-\frac{1}{3}}}{1+x^2}\mathrm{d}x \to \frac{\pi(\sqrt{3}+i)}{3+\sqrt{3}i}$$

as $\rho \to 0$ and $R \to \infty$, i.e.,

$$\int_0^\infty \frac{x^{-\frac{1}{3}}}{1+x^2}\mathrm{d}x = \frac{\pi\sqrt{3}}{3}.$$

EXAMPLES

Evaluate the integrals:

2.5.1

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2-2x+2)\sqrt{x}}.$$

ANSWER: $(\pi \sin(3\pi/8))/\sqrt[4]{2}$.

2.5.2

$$\int_0^\infty \frac{x^p\,dx}{1+2x\cos\theta+x^2},\quad 0<p<1,\quad -\pi/2<\theta<\pi/2.$$

2.6 Sixth method

This can be used for integrals

$$\int_0^\infty f(x)dx, \qquad \int_0^\infty f(x)\{\ln x\}^k dx, \qquad k=1,2,\ldots,$$

where

(i) except for at most a finite number of singular points, none of which lies on the non-negative part of the real axis, $f(z)$ is analytic on the entire complex plane and

(ii)

$$f(z)=O(z^{-2}) \qquad \text{as} \qquad z\to\infty.$$

Conditions (i) and (ii) imply that the integrals $\int_0^\infty f(x)dx$ and $\int_1^\infty f(x)\{\ln x\}^k dx$ converge absolutely. Since $\{\ln x\}^k = o(x^{-\frac{1}{2}})$ as $x\to 0+$ and $f(x)$ is continuous at 0, $\int_0^1 f(x)\{\ln x\}^k dx$ converges absolutely.

Condition (ii) can be replaced by the condition

$$\text{(A)} \qquad f(z)=O(|z|^{-\alpha}),\quad \alpha>1,$$

which appears to be weaker. However, it can be shown that (i) and (A) imply (ii), hence one might just as well use the simpler condition (ii).

Method 6 is useful for the evaluation of integrals of the form $\int_0^\infty [P(x)/Q(x)]dx$, where $P(x)$ and $Q(x)$ are polynomials such that the degree of Q exceeds the degree of P by at least two and Q has no non-negative, real zeros. Consequently, this method may be used to evaluate some of the non-even integrands, that could not be evaluated by Method 3. Method 6 cannot be used on integrands of the form $[P(x)/Q(x)]\cos x$, $[P(x)/Q(x)]\sin x$ for the same reason as in the case of Method 5. Hence, as before, no attempt will be made to generalize Method 6 to integrands with poles on the non-negative part of the real axis.

WORKED EXAMPLES.

2.6.a Evaluate the integral

$$\int_0^\infty \frac{x\,\mathrm{d}x}{1+x^6}.$$

SOLUTION: Consider the two integrals

$$\int_{C_1} f_1(z)\mathrm{d}z, \qquad \int_{C_2} f_2(z)\mathrm{d}z,$$

where C_1 and C_2 are the two contours, shown in Fig. 8,

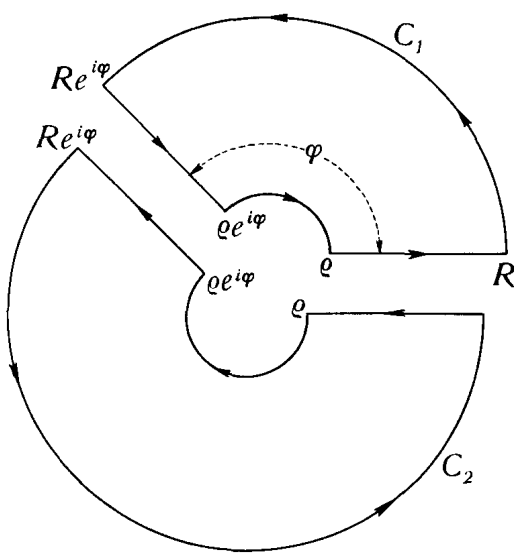

Fig. 8

$$f_1(z) = \frac{z \ln z}{1+z^6}, \qquad -\frac{\pi}{2} < \mathscr{I}m \ln z < \frac{3\pi}{2},$$

$$f_2(z) = \frac{z \ln z}{1+z^6}, \qquad \frac{\pi}{2} < \mathscr{I}m \ln z < \frac{5\pi}{2}$$

and the angle ϕ and the radii ρ, R are chosen so that all singular points of $1/(1+z^6)$ in the upper half-plane are inside and all the remaining singular points are inside C_2.

The function $f_1(z)$ has inside C_1 the singular points

$$z_r = e^{i(2r-1)\pi/6}, \qquad r = 1, 2, 3$$

and the residue at z_r is

$$\lim_{z\to z_r} \frac{z(z-z_r)\ln z}{1+z^6} = \frac{z_r \ln z_r}{6z_r^5} = \frac{\ln z_r}{6z_r^4} = \frac{i(2r-1)\pi}{36} e^{-2(2r-1)\pi i/3}.$$

The function $f_2(z)$ has inside C_2 the singular points

$$z_r = e^{i(2r-1)\pi/6}, \qquad r = 4, 5, 6$$

and the residue at z_r is

$$\frac{i(2r-1)\pi}{36} e^{-2(2r-1)\pi i/3}.$$

By the residue theorem,

$$\int_{C_1} f_1(z)\,dz + \int_{C_2} f_2(z)\,dz = 2\pi i \sum_{r=1}^{6} \frac{i(2r-1)\pi}{36} e^{-2(2r-1)\pi i/3}.$$

However, as in **2.5.a**

$$\int_{C_1} f_1(z)\,dz + \int_{C_2} f_2(z)\,dz = \int_\rho^R f_1(x)\,dx - \int_\rho^R f_2(x)\,dx + \int_{\Gamma_1} f_1(z)\,dz + \int_{\Gamma_2} f_2(z)\,dz + \int_{\gamma_1} f_1(z)\,dz + \int_{\gamma_2} f_2(z)\,dz,$$

where Γ_i is the large circular arc and γ_i the small circular arc of the contour C_i (Fig. 8). When z lies on Γ_j,

$$|f_j(z)| = \left|\frac{z\ln z}{z^6+1}\right| \leqq \frac{R\sqrt{(\ln R)^2+(\arg z)^2}}{R^6-1} \leqq \frac{R\sqrt{(\ln R)^2+4\pi^2}}{R^6-1};$$

hence

$$\left|\int_{\Gamma_j} f_j(z)\,dz\right| \leqq \frac{R\sqrt{(\ln R)^2+4\pi^2}}{R^6-1} \cdot \text{length of } \Gamma_j < \frac{2\pi R^2\sqrt{(\ln R)^2+4\pi^2}}{R^6-1}.$$

Thus,

$$\int_{\Gamma_j} f_j(z)\,dz \to 0$$

as $R \to \infty$. Since $z/(z^6+1)$ is continuous at 0, there exists a constant K such that $|z/(z^6+1)| \leqq K$ for all z in a neighbourhood of 0. Then, when ρ is small and z lies on γ_j,

$$|f_j(z)| = \left|\frac{z \ln z}{z^6+1}\right| \leqq K \sqrt{(\ln \rho)^2+4\pi^2},$$

so that

$$\left|\int_{\gamma_j} f_j(z)\mathrm{d}z\right| \leqq K\sqrt{(\ln \rho)^2+4\pi^2} \cdot \text{length of } \gamma_j$$
$$= K\sqrt{(\ln \rho)^2+4\pi^2} \cdot 2\pi\rho.$$

Thus,

$$\int_{\gamma_j} f_j(z)\mathrm{d}z \to 0$$

as $\rho \to 0$.

Thus,

$$\int_\rho^R \{f_1(x)-f_2(x)\}\mathrm{d}x \to 2\pi i \sum_{r=1}^{6} \frac{i(2r-1)\pi}{36} e^{-2(2r-1)\pi i/3}$$
$$= -\frac{\pi^2}{18} \sum_{r=1}^{6} (2r-1)e^{-2(2r-1)\pi i/3}$$

$$= -\frac{\pi^2}{18}\left\{\cos\frac{2\pi}{3} + 3\cos\frac{6\pi}{3} + 5\cos\frac{10\pi}{3} + 7\cos\frac{14\pi}{3} + 9\cos\frac{18\pi}{3} + 11\cos\frac{22\pi}{3}\right\} - i\left\{\sin\frac{2\pi}{3} + 3\sin\frac{6\pi}{3} + 5\sin\frac{10\pi}{3} + 7\sin\frac{14\pi}{3} + 9\sin\frac{18\pi}{3} + 11\sin\frac{22\pi}{3}\right\}$$

$$= -\frac{2\sqrt{3}\pi^2}{9} i$$

as $\rho \to 0$ and $R \to \infty$, i.e.,

$$\int_\rho^R \frac{x}{1+x^6}\{\ln x-(\ln x+2\pi i)\}\mathrm{d}x \to -\frac{2\sqrt{3}\pi^2}{9} i,$$

and, finally,

$$\int_\rho^R \frac{x\,dx}{1+x^6} \to \frac{\sqrt{3}\pi}{9}$$

as $\rho \to 0$ and $R \to \infty$. Hence

$$\int_0^\infty \frac{x\,dx}{1+x^6} = \frac{\pi\sqrt{3}}{9}.$$

2.6.b Evaluate

$$\int_0^\infty \frac{\ln^2 x}{(1+x)^2}\,dx.$$

SOLUTION: This integral will be evaluated by obtaining a recursion formula for the integral

$$I_k = \int_0^\infty \frac{\ln^k x}{(1+x)^2}\,dx, \qquad k = 0, 1, 2, \ldots.$$

Consider the integrals

$$\int_{C_1} f_1(z)dz, \qquad \int_{C_2} f_2(z)dz,$$

where C_1 and C_2 are the contours of Example **2.5.a**, ρ, R and the angle ϕ are such that the pole $z = -1$ of the function $1/(1+z)^2$ lies inside C_2 and

$$f_1(z) = \frac{\ln^{k+1} z}{(1+z)^2}, \quad -\frac{\pi}{2} < \mathcal{I}m \ln z < \frac{3\pi}{2},$$

$$f_2(z) = \frac{\ln^{k+1} z}{(1+z)^2}, \quad \frac{\pi}{2} < \mathcal{I}m \ln z < \frac{5\pi}{2}.$$

The residue of $f_2(z)$ at -1 is

$$\lim_{z\to -1} \frac{d}{dz}[(z+1)^2 f_2(z)] = \lim_{z\to -1} \frac{1}{z}(k+1)\ln^k z$$
$$= -(k+1)\ln^k(-1) = -(k+1)(\pi i)^k.$$

By the residue theorem,

$$\int_{C_1} f_1(z)dz + \int_{C_2} f_2(z)dz = 2\pi i[-(k+1)(\pi i)^k]$$
$$= -2(k+1)(\pi i)^{k+1}.$$

However,

$$\int_{C_1} f_1(z)\mathrm{d}z+\int_{C_2} f_2(z)\mathrm{d}z=\int_\rho^R f_1(x)\mathrm{d}x-\int_\rho^R f_2(x)\mathrm{d}x+\int_{\Gamma_1} f_1(z)\mathrm{d}z$$
$$+\int_{\Gamma_2} f_2(z)\mathrm{d}z+\int_{\gamma_1} f_1(z)\mathrm{d}z+\int_{\gamma_2} f_2(z)\mathrm{d}z.$$

But

$$\left|\int_{\Gamma_j} f_j(z)\mathrm{d}z\right| \leqq \frac{2\pi R\{(\ln R)^2+4\pi^2\}^{(k+1)/2}}{(R-1)^2},$$

when $R>1$; hence

$$\int_{\Gamma_j} f_j(z)\mathrm{d}z \to 0$$

as $R\to\infty$. Since $1/(1+z)^2$ is continuous at 0, there exists a constant K such that $\left|1/(1+z)^2\right| \leqq K$ for all z in a neighbourhood of 0. Therefore, when ρ is small,

$$\left|\int_{\gamma_j} f_j(z)\mathrm{d}z\right| \leqq 2\pi\rho K\{(\ln\rho)^2+4\pi^2\}^{(k+1)/2};$$

hence

$$\int_{\gamma_j} f_j(z)\mathrm{d}z \to 0$$

as $\rho\to 0$.

Thus,

$$\int_\rho^R \{f_1(x)-f_2(x)\}\mathrm{d}x \to -2(k+1)(\pi i)^{k+1}$$

as $\rho\to 0$ and $R\to\infty$.

Since

$$\int_\rho^R \{f_1(x)-f_2(x)\}\mathrm{d}x=\int_\rho^R \frac{1}{(1+x)^2}\left[\ln^{k+1}x-\{\ln x+2\pi i\}^{k+1}\right]\mathrm{d}x$$
$$=-\sum_{r=1}^{k+1}\binom{k+1}{r}(2\pi i)^r\int_\rho^R \frac{\ln^{k+1-r}x}{(1+x)^2}\,\mathrm{d}x,$$

one has

$$\sum_{r=1}^{k+1}\binom{k+1}{r}(2\pi i)^r I_{k+1-r}=2(k+1)(\pi i)^{k+1}.$$

By putting $k = 0$, one obtains $I_0 = 1$, while $k = 1$ yields

$$4\pi i I_1 - 4\pi^2 I_0 = -4\pi^2,$$

so that $I_1 = 0$. Finally, for $k = 2$, one obtains

$$6\pi i I_2 - 12\pi^2 I_1 - 8\pi^3 i I_0 = -6\pi^3 i;$$

hence

$$I_2 = \tfrac{1}{3}\pi^2, \quad \text{i.e.,}$$

$$\int_0^\infty \frac{\ln^2 x}{(1+x)^2}\,dx = \tfrac{1}{3}\pi^2.$$

EXAMPLES

Evaluate the integrals

2.6.1

$$\int_0^\infty \frac{x\,dx}{(x+1)^6}.$$

2.6.2

$$\int_0^\infty \frac{\sqrt{x}\ln x}{(1+x)^2}\,dx.$$

HINT: Use the substitution $x = t^2$.

ANSWER: π.

2.6.3

$$\int_0^\infty \frac{\ln^2 x}{1+x^2}\,dx.$$

ANSWER: $\pi^3/8$.

2.7 SEVENTH METHOD

This method applies to integrals of the form

$$\int_{-\infty}^{\infty} f(e^x)\{g(x) - g(x+2\pi i\}\,dx,$$

where

(i) $f(z)$ is analytic on the entire complex plane except fo

most a finite number of singular points none of which lies on the positive part of the real axis,

(ii) $g(z)$ is analytic on an open set containing the region

$$H = \{x+iy;\ -\infty < x < \infty \quad \text{and} \quad 0 \leqq y \leqq 2\pi\}$$

except for at most a finite number of singular points none of which lies on the boundary of H and

(iii) $f(e^z)g(z) \to 0$ as $z \to \infty$ in H.

Consider the integral

$$\int_C f(e^z)\,g(z)\,dz,$$

where C is the rectangular contour shown in Fig. 9.

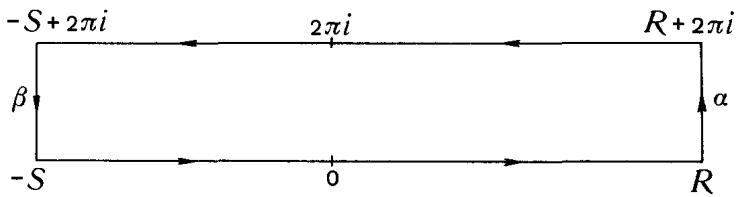

Fig. 9

Since the function e^z is 1—1 on the interior of H, the integrand $f(e^z)g(z)$ has only a finite number of singularities in the interior of H. By the residue theorem,

$$\int_C f(e^z)\,g(z)\,dz \to 2\pi i \sum_{\zeta} (\text{residue of } f(e^z)g(z) \text{ at } \zeta)$$

as $R \to \infty$ and $S \to \infty$, where the summation is taken over all singular points ζ of $f(e^z)g(z)$ in H. But

$$\int_C f(e^z)\,g(z)\,dz = \int_{-S}^{R} [f(e^x)\,g(x) - f(e^{x+2\pi i})\,g(x+2\pi i)]\,dx + \int_\alpha f(e^z)\,g(z)\,dz + \int_\beta f(e^z)\,g(z)\,dz,$$

where α and β are the two vertical parts of the contour C (Fig. 9). Hence

$$\int_{-S}^{R} f(e^x)\{g(x) - g(x+2\pi i)\}\,dx + \int_\alpha f(e^z)\,g(z)\,dz + \int_\beta f(e^z)\,g(z)\,dz$$

approaches a limit as $R \to \infty$ and $S \to \infty$. Since $f(e^z)g(z) \to 0$ as $z \to \infty$ in H, there exists for $\varepsilon > 0$ a number N such that

$$|f(e^z)g(z)| < \varepsilon$$

for all $z \in H$ with $|z| \geqq N$. Hence, when $R > N$,

$$\left| \int_\alpha f(e^z)\, g(z)\, \mathrm{d}z \right| < 2\pi\varepsilon,$$

so that

$$\int_\alpha f(e^z)\, g(z)\, \mathrm{d}z \to 0$$

as $R \to \infty$. Similarly,

$$\int_\beta f(e^z)\, g(z)\, \mathrm{d}z \to 0$$

as $S \to \infty$. Hence

$$\int_{-S}^{R} f(e^x)\{g(x) - g(x+2\pi i)\}\, \mathrm{d}x$$

approaches a limit as $R \to \infty$ and $S \to \infty$. Thus, the integral

$$\int_{-\infty}^{\infty} f(e^x)\{g(x) - g(x+2\pi i)\}\, \mathrm{d}x$$

has been shown to converge, at least conditionally. In most cases, the integral will converge absolutely.

WORKED EXAMPLES

2.7.a Show that

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+e^{2a}} \cdot \frac{1}{x^2+\pi^2}\, \mathrm{d}x = \frac{2\pi e^{-a}}{4a^2+\pi^2} - \frac{1}{1+e^{2a}},$$

where a is a real number.

SOLUTION: Clearly, the integral converges absolutely. Since

$$\frac{1}{z^2+\pi^2} = \frac{i}{2\pi(z+\pi i)} - \frac{i}{2\pi(z-\pi i)},$$

let

$$g(z) = \frac{-i}{2\pi(z-\pi i)}$$

and consider the integral

$$\int_C \frac{e^z}{e^{2z}+e^{2a}} \cdot \frac{1}{z-\pi i}\, dz,$$

where C is the contour shown in Fig. 10.

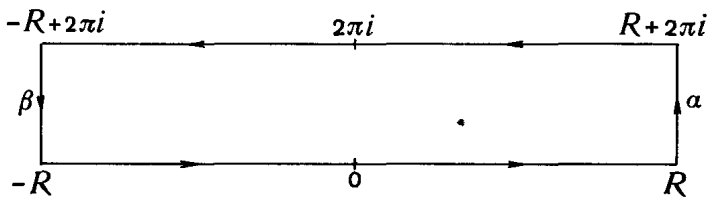

Fig. 10

The function

$$\frac{e^z}{e^{2z}+e^{2a}} \cdot \frac{1}{z-\pi i}$$

has singular points where $z = \pi i$ and $z = a+(n+\frac{1}{2})\pi i$. Of thes πi, $a+\frac{1}{2}\pi i$, $a+\frac{3}{2}\pi i$ lie inside C when $R > |a|$. The residue at πi

$$\lim_{z\to\pi i} \frac{e^z}{e^{2z}+e^{2a}} = -\frac{1}{1+e^{2a}}.$$

Let $\phi(z) = e^{2z}$. Obviously,

$$\phi'\{a+(n+\tfrac{1}{2})\pi i\} \text{ exists and}$$
$$= 2e^{2a+(2n+1)\pi i} = -2e^{2a} \neq 0$$

Hence

$$\lim_{z\to a+(n+\frac{1}{2})\pi i} \frac{z-\{a+(n+\frac{1}{2})\pi i\}}{e^{2z}-e^{2a+(2n+1)\pi i}}$$

exists and is equal to $-\frac{1}{2}e^{-2a}$, i.e.,

$$\lim_{z\to a+(n+\frac{1}{2})\pi i} \frac{z-\{a+(n+\frac{1}{2})\pi i\}}{e^{2z}+e^{2a}}$$

exists and is equal to $-\frac{1}{2}e^{-2a}$. Therefore

$$\lim_{z\to a+(n+\frac{1}{2})\pi i} \frac{e^z[z-\{a+(n+\frac{1}{2})\pi i\}]}{(e^{2z}+e^{2a})(z-\pi i)}$$

exists and

$$= -\frac{e^{a+(n+\frac{1}{2})\pi i}\frac{1}{2}e^{-2a}}{a+(n+\frac{1}{2})\pi i-\pi i} = -\frac{(-1)^n ie^{-a}}{2\{a+(n-\frac{1}{2})\pi i\}}.$$

Thus, the residue at $a+(n+\frac{1}{2})\pi i$ is $-(-1)^n ie^{-a}/2\{a+(n-\frac{1}{2})\pi i\}$; hence the residues at $a+\pi i/2$, $a+3\pi i/2$ are

$$-\frac{ie^{-a}}{2(a-\frac{1}{2}\pi i)}, \quad \frac{ie^{-a}}{2(a+\frac{1}{2}\pi i)}.$$

By the residue theorem, when $R > |a|$,

$$\int_C \frac{e^z}{e^{2z}+e^{2a}}\cdot\frac{dz}{z-\pi i} = 2\pi i\left(-\frac{1}{1+e^{2a}} - \frac{ie^{-a}}{2(a-\frac{1}{2}\pi i)} + \frac{ie^{-a}}{2(a+\frac{1}{2}\pi i)}\right)$$

$$= 2\pi i\left(\frac{2\pi e^{-a}}{4a^2+\pi^2} - \frac{1}{1+e^{2a}}\right).$$

However,

$$\int_C \frac{e^z}{e^{2z}+e^{2a}}\cdot\frac{1}{z-\pi i}\,dz$$

$$= \int_{-R}^{R}\left[\frac{e^x}{e^{2x}+e^{2a}}\cdot\frac{1}{x-\pi i} - \frac{e^{x+2\pi i}}{e^{2x+4\pi i}+e^{2a}}\cdot\frac{1}{x+\pi i}\right]dx$$

$$+\int_\alpha \frac{e^z}{e^{2z}+e^{2a}}\cdot\frac{1}{z-\pi i}\,dz + \int_\beta \frac{e^z}{e^{2z}+e^{2a}}\cdot\frac{1}{z-\pi i}\,dz$$

$$= 2\pi i\int_{-R}^{R}\frac{e^x}{e^{2x}+e^{2a}}\frac{1}{x^2+\pi^2}\,dx + \int_\alpha \frac{e^z}{e^{2z}+e^{2a}}\cdot\frac{1}{z-\pi i}\,dz$$

$$+\int_\beta \frac{e^z}{e^{2z}+e^{2a}}\cdot\frac{1}{z-\pi i}\,dz,$$

where α and β denote the vertical parts on the right-hand and left-hand sides, respectively, of the contour C. (Fig. 10).

When z lies on α and $R > a$,

$$\left| \frac{e^z}{e^{2z}+e^{2a}} \cdot \frac{1}{z-\pi i} \right| \leqq \frac{e^R}{e^{2R}-e^{2a}} \cdot \frac{1}{R};$$

hence

$$\left| \int_\alpha \frac{e^z}{e^{2z}+e^{2a}} \cdot \frac{1}{z-\pi i}\, dz \right| \leqq \frac{e^R}{e^{2R}-e^{2a}} \cdot \frac{2\pi}{R},$$

so that

$$\int_\alpha \frac{e^z}{e^{2z}+e^{2a}} \cdot \frac{1}{z-\pi}\, dz \to 0$$

as $R \to \infty$. When z lies on β and $-R < a$,

$$\left| \frac{e^z}{e^{2z}+e^{2a}} \cdot \frac{1}{z-\pi i} \right| \leqq \frac{e^{-R}}{e^{2a}-e^{-2R}} \cdot \frac{1}{R},$$

so that, similarly,

$$\int_\beta \frac{e^z}{e^{2z}+e^{2a}} \cdot \frac{1}{z-\pi}\, dz \to 0$$

as $R \to \infty$.

Thus,

$$2\pi i \int_{-R}^{R} \frac{e^x}{e^{2x}+e^{2a}} \frac{1}{x^2+\pi^2}\, dx \to 2\pi i \left(\frac{2\pi e^{-a}}{4a^2+\pi^2} - \frac{1}{1+e^{2a}} \right)$$

as $R \to \infty$, i.e.,

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+e^{2a}} \frac{1}{x^2+\pi^2}\, dx = \frac{2\pi e^{-a}}{4a^2+\pi^2} - \frac{1}{1+e^{2a}}.$$

2.7.b Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx, \qquad 0 < a <$$

SOLUTION: The integral clearly converges absolutely. Since

$$e^{az} = \frac{e^{az} - e^{az}e^{a2\pi i}}{1 - e^{a2\pi i}}$$
$$= \frac{e^{az} - e^{a(z+2\pi i)}}{1 - e^{a2\pi i}},$$

let

$$g(z) = \frac{e^{az}}{1 - e^{a2\pi i}}$$

and consider the integral

$$\int_C \frac{e^{az}}{1+e^z}\,dz,$$

where C is the contour shown in Fig. 11. The function

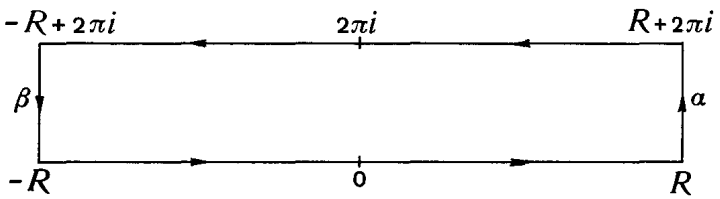

Fig. 11

$$\frac{e^{az}}{1+e^z}$$

has singular points at $z = (2n+1)\pi i$, $n = 0, \pm 1, \pm 2$, Of these only πi lies inside C. Let $\phi(z) = e^z$. Then $\phi'(\pi i)$ e and $\phi'(\pi i) = e^{\pi i} = -1$. Hence

$$\lim_{z\to\pi i} \frac{z-\pi i}{e^z - e^{\pi i}}$$

exists and is equal to -1, i.e.,

$$\lim_{z\to\pi i} \frac{z-\pi i}{1+e^z} = -1.$$

Then

$$\lim_{z\to\pi i} \frac{e^{az}(z-\pi i)}{1+e^z}$$

exists and is equal to $-e^{a\pi i}$, so that the residue at πi is $-e^{a\pi i}$.

By the residue theorem,

$$\int_C \frac{e^{az}}{1+e^z}\,dz = -2\pi i e^{a\pi i}.$$

But

$$\int_C \frac{e^{az}}{1+e^z}\,dz = \int_{-R}^{R}\left[\frac{e^{ax}}{1+e^x} - \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\right]dx + \int_\alpha \frac{e^{az}}{1+e^z}\,dz + \int_\beta \frac{e^{az}}{1+e^z}\,dz,$$

where α and β denote the vertical parts on the right-hand and left-hand sides, respectively, of the contour C (Fig. 11). When z lies on α, one has, where $z = R+iy$,

$$\left|\frac{e^{az}}{1+e^z}\right| \leqq \frac{|e^{az}|}{|e^z|-1} = \frac{e^{aR}}{e^R-1};$$

hence

$$\left|\int_\alpha \frac{e^{az}}{1+e^z}\,dz\right| \leqq \frac{2\pi e^{aR}}{e^R-1},$$

so that

$$\int_\alpha \frac{e^{az}}{1+e^z}\,dz \to 0$$

as $R \to \infty$. When z lies on β, by putting $z = -R+iy$, one obtains

$$\left|\frac{e^{az}}{1+e^z}\right| \leqq \frac{|e^{az}|}{1-|e^z|} = \frac{e^{-aR}}{1-e^{-R}};$$

hence

$$\left|\int_\beta \frac{e^{az}}{1+e^z}\,dz\right| \leqq \frac{2\pi e^{-aR}}{1-e^{-R}},$$

so that

$$\int_\beta \frac{e^{az}}{1+e^z}\,dz \to 0$$

as $R \to \infty$.

Thus,

$$(1-e^{a2\pi i})\int_{-R}^{R}\frac{e^{ax}}{1+e^{x}}\,dx \to -2\pi i e^{a\pi i}$$

as $R\to\infty$, i.e.,

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}\,dx = -\frac{2\pi i e^{a\pi i}}{1-e^{a2\pi i}}$$
$$= \frac{2\pi i}{e^{a\pi i}-e^{-a\pi i}} = \frac{\pi}{\sin \pi a}.$$

EXAMPLES.

Evaluate the integrals:

2.7.1

$$\int_{0}^{\infty}\frac{x^2}{\cosh x}\,dx.$$

HINT: Find a polynomial $P(z)$ such that, for all z,

$$P(z)-P(z+2\pi i) = z^2.$$

ANSWER: $\pi^3/8$.

2.7.2

$$\int_{0}^{\infty}\frac{x^2}{(x^2+\pi^2)\cosh x}\,dx.$$

ANSWER: $(\pi^2-3\pi)/2$.

2.7.3

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh \pi x}\,dx, \qquad -\pi < a <$$

ANSWER: sec $(a/2)$.

2.8 MISCELLANEOUS EXAMPLES

Evaluate the following integrals:

2.8.1

$$\int_{0}^{\infty}\frac{x\sin x}{x^4+1}\,dx.$$

2.8.2

$$\int_{-\infty}^{\infty} \frac{\sin m(x-a)}{x-a} \frac{\sin n(x-b)}{x-b}\, dx,$$

where m, n are integers with $m > n > 0$ and a, b are real numb with $a \neq b$.

ANSWER:

$$\pi \frac{\sin n(a-b)}{a-b}.$$

2.8.3

$$\int_0^{\infty} \frac{x^2+1}{(x^2-2x\cos a+1)(x^2-2x\cos b+1)}\, dx,$$

when $0 < a < b < \pi/2$.

ANSWER:

$$\frac{(\pi-a)\cot a-(\pi-b)\cot b}{\cos a-\cos b}.$$

2.8.4

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2(x^2+b^2)^2(x^2+c^2)^2},$$

when $a > b > c > 0$.

2.8.5

$$\int_{-\infty}^{\infty} \frac{\cos kx}{(x-a)^2+b^2}\, dx, \qquad k \text{ a natural number } b > 0.$$

ANSWER: $(\pi e^{-kb} \cos ka)/b$.

2.8.6

(i)
$$\int_0^1 \frac{x^{-3/5}(1-x)^{3/5}}{1+x^2}\, dx.$$

HINT: Make the substitution $(1-x)/x = t$.

ANSWER: $\pi(1-\cot \frac{2}{5}\pi)/2^{1/5}$.

(ii) $$\int_0^1 \frac{x^{-m/n}\,(1-x)^{m/n}}{1+x^p}\,dx,$$

where m, n $(m < n)$ and p $(\geqq 2)$ are natural numbers.

2.8.7

$$\int_0^\infty \frac{\ln x}{x^4+1}\,dx.$$

ANSWER: $-\pi^2/8\sqrt{2}$.

2.8.8

(i) $$\int_0^\infty \frac{x^6}{x^8+1}\,dx.$$

ANSWER: $(\pi\sqrt{1+\frac{1}{2}\sqrt{2}})/4$.

(ii) $$\int_0^\infty \frac{x^{2n-2}}{x^{2n}+1}\,dx,$$

where n is a positive integer.

2.8.9

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx, \qquad 0 < a < 1.$$

ANSWER: $\pi/\sin a\pi$.

2.8.10

$$\int_0^\infty \frac{dx}{1+x^6}.$$

ANSWER: $\pi/3$.

2.8.11

$$\int_0^\infty \frac{\sqrt{x}}{x^5+1}\,dx.$$

HINT: Make the substitution $x = t^2$.

ANSWER: $\pi(\sqrt{5}-1)/5$.

2.8.12

$$\int_0^\infty \frac{x^3+5x^2+4x+45}{(x^2+4)(x^2+9)^2}\,dx.$$

2.8.13

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{tz}}{z^2-1}\,dz, \qquad t \neq 0,$$

where $a > 1$ and the path of integration is a straight line.

HINT: Make the substitution $z = a+i\zeta$.

ANSWER: $\sinh t$ when $t > 0$, 0 when $t < 0$.

2.8.14

$$\int_0^\infty \frac{x^4}{(1+x^2)^4}\,dx.$$

ANSWER: $\pi/32$.

2.8.15

$$\int_{-\infty}^{\infty}\frac{\cosh ax}{\cosh \pi x}\,dx, \qquad -\pi < a < \pi.$$

ANSWER: $\sec(a/2)$.

2.8.16

$$\int_0^\infty \frac{x\sqrt{x}}{(1+x^2)^2}\,dx.$$

ANSWER: $\pi\sqrt{2}/8$.

2.8.17

$$\int_0^\infty \frac{1}{x}\,e^{\cos x}\sin(\sin x)\,dx.$$

HINT: Integrate the function $[\exp(e^{iz})]/z$ around a suita contour.

ANSWER: $\pi(e-1)/2$.

2.8.18

$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^m(x^2+b^2)^n},$$

where $a > b > 0$ and m, n are natural numbers.

2.8.19

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{2^z}{z^2}\,dz,$$

where $a > 0$ and the path of integration is a straight line.

ANSWER: $\ln 2$.

2.8.20

$$\int_0^\infty \frac{\sinh ax}{\sinh \pi x}, \qquad -\pi < a < \pi.$$

ANSWER: $\frac{1}{2}\tan(a/2)$.

2.8.21 Show that

$$\int_0^\infty \frac{(\ln x)^{2k}}{(1-x)^2}\,dx = (-1)^{k-1}\,2^{2k}\pi^{2k}B_{2k},$$

$$\int_0^1 \frac{(\ln x)^{2k-1}}{1-x}\,dx = (-1)^k\,\frac{1}{k}\,2^{2k-2}\pi^{2k}B_{2k},$$

where $B_2, B_4, B_6, \ldots$ are Bernoulli's numbers.

HINT: In the second integral, make the substitution $x =$

REMARK: These formulae are due to Th. Anghelutza, *Luc Scintifice* - Institutul Politecnic, Cluj, 1959.

2.8.22

$$\int_0^\infty \frac{x^2}{x^4+1}\,dx.$$

ANSWER: $\pi\sqrt{2}/4$.

2.8.23

$$\int_0^\infty \frac{x\sin x}{(\pi^2+x^2)(4\pi^2+x^2)}\,dx.$$

2.8.24

$$\int_0^\infty \frac{x^m}{x^n+a}\,dx,$$

where $a > 0$ and m, n are positive integers such that $n \geqq m+$

ANSWER: $\dfrac{\pi}{na^{(n-m-1)/n} \sin\left((m+1)\pi/n\right)}$.

2.8.25

$$\int_0^\infty \frac{\tanh(\pi x/2) \sin x}{1+x^2}\,dx.$$

HINT: Integrate the function $\dfrac{e^{\pi z}-1}{e^{\pi z}+1} \dfrac{e^{iz}}{1+z^2}$ around a suitable contour.

ANSWER: $\cosh e - \sinh e \ln(2 \sinh e)$.

2.8.26

$$\lim_{a\to\infty} \int_0^\infty \frac{1}{1+(a-x+1/(2x))^2}\,dx.$$

ANSWER: π.

2.8.27

$$\int_0^\infty \frac{x^2}{(a^4+x^4)^3}\,dx, \qquad a >$$

ANSWER: $5\pi\sqrt{2}/(128a^9)$.

2.8.28

$$\int_{-\infty}^\infty \frac{x^{2m}-x^{2n}}{1-x^{2p}}\,dx,$$

where m, n, p are natural numbers such that $m < n < p$.

ANSWER:

$$\frac{\pi}{p}\left\{\cot\frac{2m+1}{2p}\pi - \cot\frac{2n+1}{2p}\pi\right\}.$$

2.8.29

$$\int_0^\infty \frac{\cos ax}{1+x^2+x^4}\,dx, \qquad a >$$

ANSWER:

$$\frac{\pi}{\sqrt{3}} \sin\left(\tfrac{1}{2}a + \frac{\pi}{6}\right) e^{-a\sqrt{3}/2}.$$

2.8.30

$$\int_0^\infty \frac{\cos kx}{(x^2+a^2)(x^2+b^2)}\,dx, \qquad a > b > 0,\ k \text{ a natural number.}$$

ANSWER:

$$\frac{ae^{-kb}-be^{-ka}}{2ab(a^2-b^2)}\,\pi.$$

2.8.31

$$\int_0^\infty \frac{\cos x}{(1+x^2)^{n+1}}\,dx,$$

where n is a positive integer.

ANSWER:

$$\frac{\pi e^{-1}}{n!\,2^{2n+1}} \sum_{k=0}^{n} \frac{2^k}{k!}\,\frac{(2n-k)!}{(n-k)!}.$$

2.8.32

$$\int_{-\infty}^\infty e^{-x^2} \cos 2bx\,dx.$$

HINT: Integrate the function e^{-z^2} around a suitable contour.

ANSWER: $\sqrt{\pi}\,e^{-b^2}$.

2.8.33

$$\int_0^\infty \frac{\cos(ax^2)-\sin(ax^2)}{x^4+1}\,dx, \qquad a > 0.$$

HINT: Integrate the function $e^{iaz^2}/(z^4+1)$ around the contour bounding the region

$$\left\{z;\ 0 \leqq |z| \leqq R \qquad \text{and} \qquad 0 \leqq \arg z \leqq \frac{\pi}{2}\right\}.$$

2.8.34

$$\int_{-\infty}^\infty \frac{e^{ax/(x^2+1)}}{x^2+1} \cos\frac{a}{x^2+1}\,dx, \qquad \int_{-\infty}^\infty \frac{e^{ax/(x^2+1)}}{x^2+1} \sin\frac{a}{x^2+1}\,dx.$$

HINT: Integrate the function

$$\frac{1}{z^2+1}\, e^{a(z+i)/(z^2+1)}$$

around the semi-circular contour of Method 1.

ANSWERS: $\pi \cos(a/2)$, $\pi \sin(a/2)$.

2.8.35

$$\int_0^\infty \frac{\ln x}{(1+x^3)x^{\frac{1}{4}}}\, dx.$$

ANSWER: $-\pi^2\sqrt{2}/9$.

2.8.36

$$\int_0^\infty \frac{\ln(1+x)}{x^{a+1}}\, dx, \qquad 0 < a <$$

CHAPTER 3

RATIONAL FUNCTIONS OF COS θ AND SIN θ

3.1 This chapter is concerned with the evaluation of integrals of the form

$$\int_0^{2\pi} R(\cos\theta,\ \sin\theta)\mathrm{d}\theta, \tag{3.1}$$

where

(i) $R(x, y)$ is a rational function of x and y, i.e.,

$$R(x, y) = \frac{P(x, y)}{Q(x, y)}$$

and P, Q are polynomials;

(ii) $Q(x, y)$ does not vanish anywhere on the unit circle $\{(x, y); x^2+y^2 = 1\}$.

Any integral of this type can be evaluated by the method which will now be described.

The integral (3.1) is transformed by a change of variable to an integral

$$\oint_C f(z)\mathrm{d}z, \tag{3.2}$$

where C is the positive orientation of the unit circle and f is a rational function of the complex variable z, which has no singular points on C. The change of variable is made by substituting $\frac{1}{2}(z+1/z)$ for $\cos\theta$, $1/2i(z-1/z)$ for $\sin\theta$ and $-(i/z)\mathrm{d}z$ for $\mathrm{d}\theta$. This yields the integral

$$-i\oint_C \frac{1}{z} R\left\{\tfrac{1}{2}\left(z + \frac{1}{z}\right),\ \frac{1}{2i}\left(z - \frac{1}{z}\right)\right\}\mathrm{d}z. \tag{3.3}$$

It is easily seen that the integrand in (3.3) has no singular points on C. Also, by making the substitution $z = e^{i\theta}$ in (3.3), one can

easily show that (3.3) and (3.1) are equal. The integral (3.3) can now be evaluated by means of the residue theorem.

Although all the integrals of this chapter can be evaluated by the method just described, the work involved can be greatly reduced in some special cases if the following devices are used.

3.1.1 If $\cos n\theta$ or $\sin n\theta$ (n an integer) occurs in the integrand, it is not necessary to express them as polynomials in $\cos\theta$ and $\sin\theta$ before making the substitution $z = e^{i\theta}$. It is far simpler to substitute

$$\cos n\theta = \tfrac{1}{2}(e^{in\theta}+e^{-in\theta}) = \tfrac{1}{2}(z^n+z^{-n}),$$
$$\sin n\theta = \frac{1}{2i}(e^{in\theta}-e^{-in\theta}) = \frac{1}{2i}(z^n-z^{-n}).$$

3.1.2 If the integral is of the form

$$\int_0^{2\pi} S(\cos\theta, \sin\theta)\cos n\theta\,\mathrm{d}\theta \qquad \text{or} \qquad \int_0^{2\pi} S(\cos\theta, \sin\theta)\sin n\theta\,\mathrm{d}\theta,$$

where S is rational and real-valued and n is an integer, then evaluate the integral

$$\int_0^{2\pi} S(\cos\theta, \sin\theta)e^{in\theta}\mathrm{d}\theta$$

by the standard substitution $z = e^{i\theta}$ and equate real and imaginary parts.

3.2 Worked examples

3.2.1 Evaluate the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{3+\cos\theta+2\sin\theta}.$$

Solution: Substitute $z = e^{i\theta}$. Then

$$\cos\theta = \tfrac{1}{2}(e^{i\theta}+e^{-i\theta}) = \tfrac{1}{2}\left(z+\frac{1}{z}\right),$$
$$\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}\left(z-\frac{1}{z}\right),$$

and

$$dz = ie^{i\theta}d\theta,$$

so that

$$d\theta = -\frac{i}{z}dz.$$

Thus

$$\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta} = -i\oint_C \frac{dz}{z\{3+\frac{1}{2}(z+1/z)-i(z-1/z)\}}$$
$$= -2i\oint_C \frac{dz}{(1-2i)z^2+6z+1+2i},$$

where C is the positive orientation of the unit circle.

The function

$$f(z) = \frac{1}{(1-2i)z^2+6z+1+2i} = \frac{1}{(1-2i)\{z+(1+2i)\}\{z+\frac{1}{5}(1+2i}$$

has poles of order one at $-1-2i$ and $(-1-2i)/5$. Of these o the latter lies inside the unit circle. The residue at $(-1-2i)/$

$$\lim_{z\to -1/5(1+2i)} \frac{1}{(1-2i)\{z+(1+2i)\}} = \tfrac{1}{4}.$$

Hence, by the residue theorem,

$$\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta} = (-2i)\cdot 2\pi i\cdot \tfrac{1}{4}$$
$$= \pi.$$

3.2.2 Evaluate

$$J = \int_0^{\pi} (2\cos t)^{n+k}\cos(n-k)t\,dt,$$

where n and k are positive integers.

SOLUTION: Since

$$\{2\cos(t+\pi)\}^{n+k}\cos\{(n-k)(t+\pi)\}$$
$$= (-1)^{n+k}(-1)^{n-k}(2\cos t)^{n+k}\cos(n-k)t$$
$$= (2\cos t)^{n+k}\cos(n-k)t,$$

it follows that

$$J = \tfrac{1}{2}\int_0^{2\pi} (2\cos t)^{n+k} \cos(n-k)t \, dt.$$

Let

$$J_1 = \tfrac{1}{2}\int_0^{2\pi} (2\cos t)^{n+k} \sin(n-k)t \, dt.$$

Then

$$J + iJ_1 = \tfrac{1}{2}\int_0^{2\pi} (2\cos t)^{n+k} e^{i(n-k)t} dt.$$

Substitute $z = e^{it}$ in this last integral. Then $2\cos t = z + 1/z$ and $dz = ie^{it}\,dt = iz\,dt$; hence

$$\begin{aligned} J + iJ_1 &= -\frac{i}{2}\int_C \left(z + \frac{1}{z}\right)^{n+k} z^{n-k} \frac{1}{z}\, dz \\ &= -\frac{i}{2}\int_C (z^2+1)^{n+k} z^{-2k-1}\, dz, \end{aligned}$$

where C is the positive orientation of the unit circle.

The integrand has a pole of order $2k+1$ at $z = 0$. When $(1+z^2)^{n+k}$ is expanded by the binomial theorem, the coefficient of z^{2k} is

$$\binom{n+k}{k}.$$

This is the residue of $(1+z^2)^{n+k} z^{-2k-1}$ at 0. Then, by the residue theorem,

$$J + iJ_1 = -\frac{i}{2} \cdot 2\pi i \binom{n+k}{k} = \pi \binom{n+k}{k}.$$

By equating real parts, one finds

$$J = \pi \binom{n+k}{k}.$$

3.2.3 Evaluate the integral

$$I = \int_0^{\pi} \frac{\sin^2 x}{(1-2a\cos x + a^2)(1-2b\cos x + b^2)}\, dx, \qquad 0 < a < b < 1.$$

SOLUTION: Since $\cos(-x) = \cos x$ and $\sin^2(-x) = \sin^2 x$, one has

$$2I = \int_{-\pi}^{\pi} \frac{\sin^2 x}{(1-2a\cos x+a^2)(1-2b\cos x+b^2)}\,dx.$$

Substitute $z = e^{ix}$ in this last integral. Then $2\cos x = z + 1/z$, $\sin^2 x = -\frac{1}{4}(z-1/z)^2$ and $dz = ie^{ix}dx = iz\,dx$; hence

$$\begin{aligned} 2I &= \frac{i}{4}\int_C \frac{(z-1/z)^2}{\{1-a(z+1/z)+a^2\}\{1-b(z+1/z)+b^2\}z}\,dz \\ &= \frac{i}{4}\int_C \frac{(z^2-1)^2}{z(az-1)(z-a)(bz-1)(z-b)}\,dz, \end{aligned}$$

where C is the positive orientation of the unit circle. The integrand has singular points where $z = 0,\ 1/a,\ a,\ 1/b,\ b$. Since $0 < a < b < 1$, these five numbers are all distinct, and hence there is a pole of order one at each point.

The residue at 0 is

$$\lim_{z\to 0} \frac{(z^2-1)^2}{(az-1)(z-a)(bz-1)(z-b)} = \frac{1}{ab}.$$

The residue at a is

$$\lim_{z\to a} \frac{(z^2-1)^2}{z(az-1)(bz-1)(z-b)} = \frac{a^2-1}{a(ab-1)(a-b)},$$

and, by symmetry, the residue at b is

$$\frac{b^2-1}{b(ab-1)(b-a)}.$$

Since only the singular points at 0, a and b lie inside C,

$$\begin{aligned} I &= \frac{i}{8}\cdot 2\pi i\cdot\left[\frac{1}{ab} + \frac{a^2-1}{a(ab-1)(a-b)} + \frac{b^2-1}{b(ab-1)(b-a)}\right] \\ &= \frac{\pi}{2(1-ab)}. \end{aligned}$$

3.3 Examples

Evaluate the following integrals:

3.3.1

$$\text{(i)} \quad \int_0^{\pi/2} \frac{1}{1+\sin^2 x}\,dx, \qquad \text{(ii)} \quad \int_0^{2\pi} \frac{1-\cos 2x}{2+\cos x}\,dx,$$

$$\text{(iii)} \quad \int_0^{2\pi} \frac{\cos^2 3x}{1-2a\cos 2x+a^2}\,dx, \qquad |a|<1.$$

Answers:

(i) $\pi\sqrt{2}/4$, (ii) $8\pi(1-(\sqrt{3}/2))$, (iii) $\pi(1-a+a^2)/(1-a)$

3.3.2

$$\int_0^{\pi} \sin^{2n} x\,dx,$$

where n is a positive integer.

Answer: $\pi(2n)!/(2^{2n}(n!)^2)$.

3.3.3

$$\text{(i)} \quad \int_0^{\pi} \frac{\cos 3x}{2+\cos x}\,dx, \qquad \text{(ii)} \quad \int_0^{\pi} \frac{dx}{(4\cos x+5)^2},$$

$$\text{(iii)} \quad \int_0^{2\pi} \frac{1+\cos x}{1+\cos^2 x}\,dx, \qquad \text{(iv)} \quad \int_0^{2\pi} \frac{1+\cos x}{(13-5\cos x)^2}\,dx,$$

$$\text{(v)} \quad \int_0^{\pi} \frac{1}{5+4\cos x}\,dx.$$

Answers:

(i) $\pi(15-26/\sqrt{3})$, (ii) $5\pi/27$, (iii) $\pi\sqrt{2}$, (iv) $\pi/48$, (v) π

3.3.4

$$J=\int_0^{2\pi} \frac{\sin^2 x}{a-b\cos x}\,dx, \qquad a>b>0.$$

Show that

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{a-bx}\,dx = \tfrac{1}{2}J.$$

ANSWER: $J = 2\pi(a-\sqrt{a^2-b^2})/b^2$.

3.3.5

$$\int_0^{\pi} \frac{\cos\theta(1+\cos\theta)}{(1+\cos^2\theta)^2}\,\mathrm{d}\theta.$$

3.3.6

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{a^2\sin^2 x+b^2\cos^2 x+c^2},$$

where a, b and c are positive real numbers.

ANSWER:

$$\frac{\pi}{2\{(b^2+c^2)(c^2+a^2)\}^{\frac{1}{2}}}.$$

3.3.7

$$\oint \frac{|\mathrm{d}z|}{|z-a|^2},$$

where C is the positive orientation of the circle $\{z; |z| = r\}$ a is a complex constant with $|a| < r$.

3.3.8

$$\int_0^{2\pi} \frac{(1+2\cos t)^n \cos nt}{1-a-2a\cos t}\,\mathrm{d}t,$$

where n is a non-negative integer and a is a real number that $-1 < a < \frac{1}{3}$.

ANSWER:

$$\frac{2\pi}{\sqrt{1-2a-3a^2}}\left(\frac{1-a-\sqrt{1-2a-3a^2}}{2a^2}\right)^n,$$

when $a \neq 0$, and 2π when $a = 0$.

3.3.9

$$\int_0^{\pi/2} \cos^n x \cos nx\,\mathrm{d}x,$$

where n is a non-negative integer.

3.3.10

$$\int_0^{\pi} \cos^n x \cos rx \, dx,$$

where n and r are non-negative integers.

3.3.11

$$\int_0^{\pi/2} \cos^{2n} x \, dx,$$

where n is a positive integer.

ANSWER:

$$\frac{(2n-1)(2n-3)\ldots 1}{2n(2n-2)\ldots 2} \cdot \frac{\pi}{2}.$$

3.3.12

$$\int_0^{2\pi} \frac{dx}{(a\cos x + b\sin x + c)^2},$$

where a, b and c are positive real numbers such that $c^2 > a^2+b^2$.

ANSWER:

$$\frac{2\pi c}{(c^2-a^2-b^2)^{3/2}}.$$

3.3.13

$$\int_0^{\pi} \sin^m x \cos(2n+1)x \, dx,$$

where m and n are non-negative integers.

3.3.14

$$\int_0^{\pi} \frac{\cos(n-1)x}{(1-2a\cos x + a^2)^n} \, dx,$$

where n is a positive integer and a is a real number such that $0 < a < 1$.

ANSWER:

$$\binom{2n-2}{n-1} \frac{\pi a^{n-1}}{(1-a^2)^{2n-1}}.$$

3.3.15

$$\int_{-\pi}^{\pi} \frac{\cos m\,x}{(1-2a\,\cos x+a^2)^n}\,\mathrm{d}x,$$

where m and n are positive integers and a is a real number su that $0 < a < 1$.

ANSWER:

$$\frac{2\pi a^{m+2n-2}}{(1-a^2)^{2n-1}} \sum_{\nu=0}^{n-1} \binom{m+n-1}{\nu} \binom{2n-\nu-2}{n-1} \left(\frac{1-a^2}{a^2}\right)^{\nu}.$$

3.3.16

$$\int_0^{\pi} (\cos mx)\ln(1-2a\,\cos x+a^2)\mathrm{d}x,$$

where m is a positive integer and a is a real number su that $0 < a < 1$.

3.3.17

$$\int_0^{\pi} (\sin mx)\ln(1-2a\,\cos x+a^2)\mathrm{d}x,$$

where m is a positive integer and a is a real number su that $0 < a < 1$.

3.3.18

$$\int_0^{2\pi} \frac{\mathrm{d}t}{a-ib\,\cos t},$$

where a and b are real numbers, with $a \neq 0$.

ANSWER:

$$\frac{2\pi}{\sqrt{a^2+b^2}} \text{ when } a > 0, \quad \frac{-2\pi}{\sqrt{a^2+b^2}} \text{ when } a < 0.$$

3.3.19

$$\int_{-1}^{1} \frac{x^{2n}}{(1-x^2)^{\frac{1}{2}}}\,\mathrm{d}x,$$

where n is a positive integer.

ANSWER:

$$\frac{(2n-1)(2n-3)\ldots 1}{2n(2n-2)\ldots 2}\pi.$$

3.3.20

$$\int_0^{2\pi} \ln(\tfrac{5}{4}+\sin x)\mathrm{d}x.$$

ANSWER: 0.

3.3.21 Let $F(z)$ satisfy the following conditions:

(i) $F(z)$ is analytic on an open set containing the disc

$$\{z;\ |z| \leqq 1\};$$

(ii) $F(z)$ is real when z is real;

(iii) $F(z)/z \to 0$ as $z \to 0$.

If $f(x, y) = \mathcal{I}m\{F(x+iy)\}$ for real x and y and if a is a number such that $|a| < 1$, then show that

$$\int_0^{2\pi} \frac{a \sin\theta}{1-2a\cos\theta+a^2} f(\cos\theta, \sin\theta)\mathrm{d}\theta = \pi F(a).$$

3.3.22

$$\int_0^{2\pi} e^{a\cos x}\cos(a\sin x - nx)\mathrm{d}x,$$

where n is a positive integer and a is a real number.

HINT: Substitute $z = e^{ix}$ into the integral

$$\int_0^{2\pi} e^{a\cos x}\cos(a\sin x - nx)\mathrm{d}x + i\int_0^{2\pi} e^{a\cos x}\sin(a\sin x - nx)\mathrm{d}x$$

ANSWER: $2\pi a^n/n!$.

3.3.23

$$\int_0^{2\pi} e^{\cos x}\cos(\sin x)\cos nx\ \mathrm{d}x,$$

where n is a positive integer.

3.3.24

$$\int_0^{\pi/2} \ln(a^2\sin^2 x+b^2\cos^2 x)\,dx,$$

where a and b are real and $0 < a < b$.

3.3.25

$$\int_0^{\pi} \frac{\cos nx}{(\cos x+\cosh a)^2}\,dx,$$

where $a \neq 0$ is a real number and n is a positive integer.

ANSWER: $(-1)^n(\pi e^{-na}/\sinh^3 a)\ (n \sinh a+\cosh a)$.

3.3.26

$$J = \int_{-\pi}^{\pi} (\cos nx)Q\ (\cos x)\,dx,$$

where Q is a polynomial of degree r.

CHAPTER 4

SUMMATION OF SERIES

4.1 This chapter is concerned with the summation of series of the form

$$\sum_{n=-\infty}^{\infty} f(n) \qquad \text{or} \qquad \sum_{n=-\infty}^{\infty} (-1)^n f(n),$$

where

(i) except for at most a finite number of singular points, $f(z)$ is analytic on the entire complex plane,

(ii) $f(z) = O(z^{-2})$ as $z \to \infty$.

As in **2.6**, condition (ii) can be replaced by the condition

$$\text{(A)} \qquad f(z) = O(|z|^{-a}), \qquad a > 1,$$

which appears to be weaker. However, it can be shown that (i) and (A) imply (ii); hence one might just as well use the simpler condition (ii).

If $f(n)$ is even with respect to n, the method will give the sum of

$$\sum_{n=0}^{\infty} f(n) \qquad \text{or} \qquad \sum_{n=0}^{\infty} (-1)^n f(n).$$

The method consists of using the residue theorem to evaluate

$$\int_{C_n} f(z)\cot \pi z \, dz \qquad \text{or} \qquad \int_{C_n} f(z)\operatorname{cosec} \pi z \, dz,$$

where C_n is the positive orientation of the square with vertices at $(n+\frac{1}{2})(\pm 1 \pm i)$ and n is a non-negative integer (Fig. 12). One then allows n to approach infinity and shows that the integral around C_n tends to zero. To do this one needs the following inequalities:

$$|\cot \pi z| < 2 \tag{4.1.1}$$

for all z on C_n;

$$|\operatorname{cosec} \pi z| \leqq 1 \tag{4}$$

for all z on C_n.

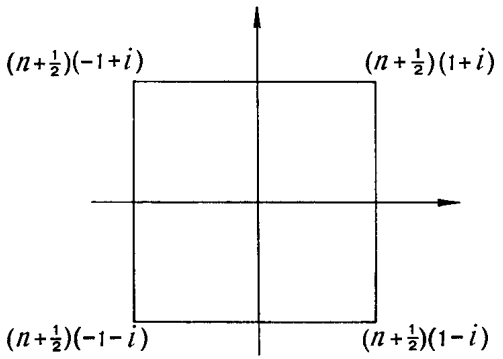

Fig. 12

To prove the first of these inequalities, observe that, by **0**

$$|\cot \pi z|^2 = \frac{\cos^2 \pi x + \sinh^2 \pi y}{\sin^2 \pi x + \sinh^2 \pi y}.$$

Hence, when z lies on a vertical side of C_n,

$$\begin{aligned} |\cot \pi z|^2 &= \frac{\cos^2 (n+\frac{1}{2})\pi + \sinh^2 \pi y}{\sin^2 (n+\frac{1}{2})\pi + \sinh^2 \pi y} \\ &= \frac{\sinh^2 \pi y}{1+\sinh^2 \pi y} < 1. \end{aligned}$$

When z lies on a horizontal side of C_n,

$$\begin{aligned} |\cot \pi z|^2 &= \frac{\cos^2 \pi x + \sinh^2 (n+\frac{1}{2})\pi}{\sin^2 \pi x + \sinh^2 (n+\frac{1}{2})\pi} \\ &\leqq \frac{\sinh^2 (n+\frac{1}{2})\pi + 1}{\sinh^2 (n+\frac{1}{2})\pi - 1} \\ &= 1 + \frac{2}{\sinh^2 (n+\frac{1}{2})\pi - 1} \\ &\leqq 1 + \frac{2}{\sinh^2 \frac{1}{2}\pi - 1} < 4. \end{aligned}$$

To prove the second inequality, observe that

$$|\operatorname{cosec} \pi z|^2 = \frac{1}{|\sin \pi z|^2}$$

and, by **0.3.2**,

$$|\operatorname{cosec} \pi z|^2 = \frac{1}{\sin^2 \pi x + \sinh^2 \pi y}.$$

Hence, when z lies on a vertical side of C_n,

$$\begin{aligned} |\operatorname{cosec} \pi z|^2 &= \frac{1}{\sin^2(n+\tfrac{1}{2})\pi + \sinh^2 \pi y} \\ &= \frac{1}{1+\sinh^2 \pi y} \leqq 1. \end{aligned}$$

When z lies on a horizontal side of C_n,

$$\begin{aligned} |\operatorname{cosec} \pi z|^2 &= \frac{1}{\sin^2 \pi x + \sinh^2(n+\tfrac{1}{2})\pi} \\ &\leqq \frac{1}{\sinh^2(n+\tfrac{1}{2})\pi - 1} \\ &\leqq \frac{1}{\sinh^2 \tfrac{1}{2}\pi - 1} < 1. \end{aligned}$$

4.2 Worked examples

4.2.1 Calculate the sum of the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2},$$

where a is real and is not an integer.

Solution: Consider the integral

$$\int_{C_n} \frac{\cot \pi z}{(a+z)^2}\, dz,$$

where n is a non-negative integer and C_n is the positive orienta of the square with vertices at $(n+\frac{1}{2})(\pm 1 \pm i)$ (Fig. 13).

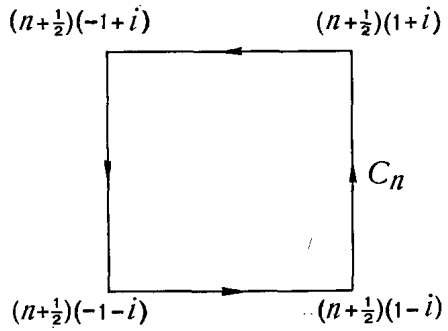

Fig. 13

The integrand has a pole of order two at $z = -a$. The res there is

$$\lim_{z\to a} \frac{\mathrm{d}}{\mathrm{d}z} (\cot \pi z) = \lim_{z\to a} (-\pi \operatorname{cosec}^2 \pi z) = -\pi \operatorname{cosec}^2 \pi a.$$

Since

$$\frac{\cot \pi z}{(a+z)^2} = \frac{\cos \pi z}{(a+z)^2 \sin \pi z},$$

there are poles of order one at $z = 0, \pm 1, \pm 2, \ldots$.

The residue at $z = m$ is

$$\lim_{z\to m} \frac{(z-m) \cos \pi z}{\sin \pi z (a+z)^2}.$$

But

$$\lim_{z\to m} \frac{\sin \pi z}{(z-m)} = \lim_{z\to m} \frac{\sin \pi z - \sin \pi m}{z-m}$$

$$= \left[\frac{\mathrm{d}}{\mathrm{d}z} (\sin \pi z)\right]_{z=m} = \pi \cos \pi m.$$

Hence, the residue at $z = m$ is

$$\frac{\cos \pi m}{\pi \cos \pi m (a+m)^2} = \frac{1}{\pi (a+m)^2}.$$

By the residue theorem

$$\int_{C_n} \frac{\cot \pi z}{(a+z)^2}\,dz = 2\pi i\left\{-\pi \operatorname{cosec}^2 \pi a + \frac{1}{\pi}\sum_{m=-n}^{n} \frac{1}{(a+m)^2}\right\},$$

when $n > |a|$. However, when z lies on C_n, it follows from (4 that

$$\left|\frac{\cot \pi z}{(a+z)^2}\right| < \frac{2}{(|z|-|a|)^2} \leqq \frac{2}{(n+\frac{1}{2}-|a|)^2}.$$

Hence

$$\left|\int_{C_n} \frac{\cot \pi z}{(a+z)^2}\,dz\right| \leqq \frac{2}{(n+\frac{1}{2}-|a|)^2} \cdot \text{length of } C_n$$

$$= \frac{2}{(n+\frac{1}{2}-|a|)^2} \cdot 4(2n+1) \to 0$$

as $n \to \infty$. Thus

$$-\pi \operatorname{cosec}^2 \pi a + \frac{1}{\pi}\sum_{m=-n}^{n} \frac{1}{(a+m)^2} \to 0$$

as $n \to \infty$, i.e.,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \pi^2 \operatorname{cosec}^2 \pi a.$$

By putting $a = \frac{1}{2}$, one finds

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^2} = \pi^2,$$

i.e.,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{4},$$

$$\sum_{n=-\infty}^{-1} \frac{1}{(2n+1)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{4},$$

$$2\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{4},$$

so that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Then

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n)^2}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$=\tfrac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{\pi^2}{8},$$

so that

$$\tfrac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

4.2.2 Calculate

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+a^2},$$

where a is real and positive.

SOLUTION: Consider the integral

$$\int_{C_n}\frac{\operatorname{cosec}\pi z}{z^2+a^2}\,\mathrm{d}z,$$

where n is a non-negative integer and C_n is the positive orienta of the square with vertices at $(n+\frac{1}{2})(\pm1\pm i)$ (Fig. 13).

The integrand has poles of order one at $ia, -ia$, 0, ±1, ±2, The residue at ia is

$$\lim_{z\to ia}\frac{(z-ia)\operatorname{cosec}\pi z}{z^2+a^2}=\lim_{z\to ia}\frac{\operatorname{cosec}\pi z}{z+ia}=\frac{\operatorname{cosec}\pi ia}{2ia}$$

$$=-\frac{1}{2a\sinh\pi a}.$$

The residue at $-ia$ is

$$\lim_{z\to -ia}\frac{(z+ia)\operatorname{cosec}\pi z}{z^2+a^2}=\lim_{z\to -ia}\frac{\operatorname{cosec}\pi z}{z-ia}=\frac{\operatorname{cosec}\pi ia}{2ia}$$

$$=-\frac{1}{2a\sinh\pi a}.$$

The residue at m is

$$\lim_{z\to m}\frac{z-m}{(z^2+a^2)\sin \pi z}.$$

But

$$\lim_{z\to m}\frac{\sin \pi z}{z-m}=\lim_{z\to m}\frac{\sin \pi z-\sin \pi m}{z-m}$$

$$=\left[\frac{\mathrm{d}}{\mathrm{d}z}(\sin \pi z)\right]_{z=m}=\pi \cos \pi m.$$

Hence, the residue at m is

$$\frac{1}{(m^2+a^2)\pi \cos \pi m}=\frac{(-1)^m}{\pi(m^2+a^2)}.$$

By the residue theorem,

$$\int_{C_n}\frac{\operatorname{cosec} \pi z}{z^2+a^2}\,\mathrm{d}z=2\pi i\left\{-\frac{1}{a\sinh \pi a}+\frac{1}{\pi}\sum_{m=-n}^{n}\frac{(-1)^m}{m^2+a^2}\right\},$$

when $n>a$. But when z lies on C_n, it follows from (4.2 that

$$\left|\frac{\operatorname{cosec} \pi z}{z^2+a^2}\right|\leqq\frac{1}{|z|^2-a^2}\leqq\frac{1}{(n+\frac{1}{2})^2-a^2}.$$

Hence

$$\left|\int_{C_n}\frac{\operatorname{cosec} \pi z}{z^2+a^2}\,\mathrm{d}z\right|\leqq\frac{1}{(n+\frac{1}{2})^2-a^2}\cdot\text{length of } C_n$$

$$=\frac{4(2n+1)}{(n+\frac{1}{2})^2-a^2}\to 0$$

as $n\to\infty$. Thus,

$$-\frac{1}{a\sinh \pi a}+\frac{1}{\pi}\sum_{m=-n}^{n}\frac{(-1)^m}{m^2+a^2}\to 0$$

as $n\to\infty$, i.e.,

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+a^2}=\frac{\pi}{a\sinh \pi a}.$$

Then

$$2\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+a^2}=\frac{\pi}{a\sinh\pi a}+\frac{1}{a^2},$$

so that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+a^2}=\frac{\pi}{2a\sinh\pi a}+\frac{1}{2a^2}.$$

4.3 Examples

Calculate the sums of the series:

4.3.1

$$\sum_{n=-\infty}^{\infty}\frac{1}{(a+n)^4},$$

where a is real and not an integer. Hence show that

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

4.3.2

(i)
$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(a+n)^2}.$$

(ii)
$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

4.3.3

(i) $$\sum_{n=1}^{\infty}\frac{1}{(n^2+4)^2},$$ (ii) $$\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+1)^3}.$$

4.3.4

$$\sum_{n=1}^{\infty}\frac{1}{n^{2l}+a^{2l}},$$

where a is real and positive and l a positive integer.

Answer:

$$\frac{\pi}{4la^{2l-1}}\sum_{k=-l}^{l-1}e^{((2k+1)/2l)\pi i}\cot\left(\pi a e^{((2k+1)/2l)\pi i}\right)-\tfrac{1}{2}a^{-2l}.$$